The average theorem

In a region with no charges, where
\[ \nabla^2 V = 0 , \] (1)
the potential at any point \( \vec{r}_0 \) is identical to the average of the potential on a sphere centered at \( r_0 \), with arbitrary radius \( R \):
\[ V(\vec{r}_0) = \frac{1}{4\pi R^2} \int_{\text{sphere}(\vec{r}_0, R)} V(\vec{r}) \, da \] (2)

Physicist’s proof:
Evidently the property (2) must hold for an infinitesimally small sphere centered at \( \vec{r}_0 \). If \( V \) is continuous, its value at \( \vec{r}_0 \) is the same as the values it takes in an infinitesimally small region around \( \vec{r}_0 \). Now, if we prove that the r.h.s. of eq. (2) in fact does not depend on \( R \), we are done: if the theorem holds for \( R \to 0 \), and the r.h.s. is \( R \)-independent, the theorem holds for all \( R \)’s. Without loss of generality, we can choose the origin of our coordinates to be \( \vec{r}_0 \) itself. This simplifies the notation somewhat.

Suppose we slightly vary \( R \):
\[ R \to R + \delta R . \] (3)

How does the r.h.s. change? We are computing an integral over a slightly larger sphere than the original one. So, first of all the area element \( da \) is affected by the change in \( R \). However since we are also dividing by the total area \( 4\pi R^2 \), the combination of these two factors is unchanged. That is, the quantity
\[ \frac{da}{4\pi R^2} = \frac{1}{4\pi} \sin \theta d\theta d\phi \] (4)
is manifestly \( R \)-independent, because it only depends on the angular variables on the sphere—not on its radius. The other thing that can change by varying \( R \) is the values the potential itself takes: on a slightly larger sphere, the potential is slightly different. By how much? Starting from any given point \( \vec{r} \) on the original sphere, we draw a radial outgoing infinitesimal vector of length \( \delta R \). This way we “generate” the new sphere. For any \( \vec{r} \), the potential at the slightly displaced point defined this way is
\[ V(\vec{r} + \hat{r} \delta R) \equiv V(\vec{r}) + \delta R \hat{r} \cdot \nabla V(\vec{r}) \] (5)
(this is one of the fundamental properties of the gradient of a function). Therefore, the variation of the r.h.s. of eq. (2) under (3) is
\[ \delta \left[ \int_{\text{sphere}(\vec{r}_0, R)} V(\vec{r}) \frac{da}{4\pi R^2} \right] = \delta R \int_{\text{sphere}(\vec{r}_0, R)} \nabla V(\vec{r}) \cdot \hat{r} \frac{da}{4\pi R^2} \] (6)

For a sphere the unit radial vector \( \hat{r} \) is the same as the normal unit vector \( \hat{n} \), so that \( \hat{r} da = d\hat{a} \). Therefore we get
\[ \frac{\delta R}{4\pi R^2} \int_{\text{sphere}(\vec{r}_0, R)} \nabla V(\vec{r}) \cdot d\hat{a} = \frac{\delta R}{4\pi R^2} \int_{\text{volume}} \nabla^2 V(\vec{r}) \, d\tau = 0 , \] (7)
where we used Gauss’s theorem to express the surface integral as the volume integral of a divergence, and we used that \( V \) obeys Laplace’s equation (1) in the region of interest. In conclusion, eq. (2) holds for \( R \to 0 \), and its r.h.s. in independent of \( R \), therefore it holds for any \( R \). As you see, a lot of talking, and few formulas. □

Mathematician’s proof:
Let’s choose the origin 0 to be \( \vec{r}_0 \). Also, to simplify the notation we will drop the subscript ‘sphere(\( \vec{r}_0, R \))’ from the integral. Instead, \( S \) will be the sphere’s surface, and \( V \), the sphere’s volume. We will make extensive use of

1. Laplace’s equation (1);
2. Gauss’s theorem to express volume integrals as surface integrals and vice-versa;
3. \( \vec{\nabla} \cdot (f \vec{g}) = \vec{\nabla}f \cdot \vec{g} + f \vec{\nabla} \cdot \vec{g} \), to “integrate by parts”;
4. the fact that for a sphere, \( r \equiv |\vec{r}| \) is constant throughout the surface, and equal to \( R \), so that we can freely factor it out (or in) of surface integrals.
5. the fact that for a sphere, \( \hat{r} \) is the same as the outgoing normal \( \hat{n} \);
6. the properties of the Dirac delta-function.

Keeping all this in mind, starting from the r.h.s. of eq. (2) we have

\[
\frac{1}{4\pi R^2} \oint_S V \, d\vec{a} = \frac{1}{4\pi R^2} \oint_S V \hat{r} \cdot d\vec{a}
\]
(8)
\[
= \frac{1}{4\pi} \int_S V \hat{r} \cdot d\vec{a}
\]
(9)
\[
= \frac{1}{4\pi} \int_V \nabla \cdot \left( V \frac{\hat{r}}{r^2} \right) \, d\tau
\]
(10)
\[
= \frac{1}{4\pi} \int_V \nabla V \cdot \frac{\hat{r}}{r^2} \, d\tau + \frac{1}{4\pi} \int_V V \nabla \cdot \frac{\hat{r}}{r^2} \, d\tau
\]
(11)
\[
= -\frac{1}{4\pi} \int_V \nabla V \cdot \frac{1}{r} \, d\tau + \int_V V \frac{1}{r} \, d\tau + V(3) \delta^3(\vec{r}) \, d\tau
\]
(12)
\[
= -\frac{1}{4\pi} \int_V \nabla \cdot \left( V \frac{1}{r} \right) \, d\tau + \frac{1}{4\pi} \int_V \nabla^2 V \frac{1}{r} \, d\tau + V(0)
\]
(13)
\[
= -\frac{1}{4\pi} \oint_S \frac{1}{r} \nabla V \cdot d\vec{a} + V(0)
\]
(14)
\[
= -\frac{1}{4\pi R} \oint_S \nabla V \cdot d\vec{a} + V(0)
\]
(15)
\[
= -\frac{1}{4\pi R} \int_V \nabla^2 V \, d\tau + V(0) = V(0)
\]
(16)

which is precisely the potential at the center of the sphere. □