a) Following the hint, approximate the larger loop by a dipole whose field $B$ along the dipole axis is given by eq. 30-29:

$$B(z) = \frac{\mu_0}{2\pi} \frac{\hat{M}}{z^3}$$

with $\hat{M} = i\pi R^2 A$ ($\hat{M} = \mu_0 R^2$)

and in our problem $z$ becomes $x$. Also then assume $\hat{M}$ constant over entire area of small loop.

So by def. of flux $\Phi$:

$$\Phi_{\text{small loop}} = \oint \hat{B} \cdot d\hat{A} = 8A = \mu_0 \left( \frac{\pi R^2}{2\pi} \right) \left( \frac{\pi R^2}{2\pi} \right)$$

$$= \frac{M_0 \pi R^2}{2}$$

b) So then by Faraday's law:

$$\mathcal{E} = -\frac{d\Phi}{dt} =$$
and by the chain rule of calculus:

\[ \frac{d}{dt} = -\frac{d\Phi}{dx} = -\frac{d\Phi}{dx} \frac{dx}{dt} = \frac{\partial \Phi}{\partial x} \]

\[ = -\left(-3 \frac{\mu_0 T R^2 r^2}{2} \right) v \]

\[ = \frac{\mu_0 3 \pi R^2 r^2 v}{2 \times 4} \]

b) direction from Lenz's Law:
- direction of \( \mathbf{B} \) from big loop by RHR: up \( \uparrow \)
- \( \mathbf{B}_0 \) direction of \( "\mathbf{B} \) for flux" in little loop: up \( \uparrow \)
As little loop moves up, \( \mathbf{B} \) used to calc. flux ("\( \mathbf{B} \) for flux") gets smaller

\[ \uparrow \mathbf{B}_{swt} \quad \uparrow \mathbf{B}_{later} \quad \uparrow \]

\[ \text{start} \quad \text{later} \quad \text{even later} \]
so $\Delta \mathbf{B}$, change in $\mathbf{B}$ for flux,
points DOWN:

\[
\begin{align*}
\Delta \mathbf{B} & \quad \text{later} \\
\mathbf{B}_{\text{late}} & \quad \mathbf{B}_{\text{late}} + \Delta \mathbf{B} \\
\mathbf{B}_{\text{even later}} &
\end{align*}
\]

Leuiz's law says $\mathbf{B}_{\text{induced}}$ should oppose this change (i.e., this $\Delta \mathbf{B}$) so we want $\mathbf{B}_{\text{induced}}$ (the field caused by the induced current) to be up $\uparrow$

and by $\text{RHR}$:

$\mathbf{B}_{\text{induced}}$
So the current should be in some direction as large loop.

31-177

a) \quad \text{SIDE VIEWS:}

\[ \Phi = N \oint \mathbf{B} \cdot d\mathbf{A} = N \mathbf{B} \cdot \mathbf{A} = N B A \cos \theta \]

\[ = N B a b \cos \theta \]

We are told, in effect, that \( \theta = 2\pi f t \) (rotating smoothly; \( \theta = \omega t = 2\pi f t \)) so:

\[ \Phi(t) = B a b \cos 2\pi f t \]

and by Faraday's law:

\[ E = -\frac{d\Phi}{dt} = N B a b (2\pi f) \sin(2\pi f t) \]

b) \( E = 2\pi 60 (N a b) (5) \) so \( a b N = 188.4 \text{ m}^2 \).
Along horizontal strips of width $dy$, $B$ is constant and $B(y) = \frac{\mu_0 i}{2\pi y}$

so $d\Phi = B \cdot d\hat{A} = B \cdot dy = \frac{\mu_0 i}{2\pi} dy$

and then

$$\Phi = \int_{r-b/2}^{r+b/2} \frac{\mu_0 i a \, dy}{2\pi y}$$

$$= \frac{\mu_0 i a}{2\pi} \ln \left( \frac{r+b/2}{r-b/2} \right)$$

b) Since loop is moving, $r$ is a function of $t$: $r(t)$ and $\frac{dr}{dt} = \dot{r}$.

Part a) gives us $\Phi(r)$ by Faraday's law

$$\varepsilon = -\frac{d\Phi}{dt} = \text{chain rule of calculus} \Rightarrow \frac{\varepsilon}{\dot{r}} = -\frac{d\Phi}{dr} \left( \frac{dr}{dt} \right)$$
\[
\mathcal{E} = -\frac{\mu_0 I V}{2\pi}\left(\frac{1}{r+b/2} - \frac{1}{r-b/2}\right)
\]

And by Ohm's law, \(\mathcal{E} = IR\),

\[
I_{loop} = \frac{M_0 V}{2\pi R}\left(\frac{1}{r-b/2} - \frac{1}{r+b/2}\right)
\]

31-28)

Call vertical height of loop aq above \(Y\). At terminal velocity \(V_T\) this height is changing at a rate \(\frac{dy}{dt} = V_T\).

The flux in the loop,

\[
\Phi = \int B dA = B A = B L y
\]

is then changing at rate

\[
\frac{d\Phi}{dt} = B L \frac{dy}{dt} = B L V_T
\]
And by Faraday's 1st law, the current is:

\[ i = \frac{\varepsilon}{R} = -\frac{1}{R} \frac{d\Phi}{dt} = -\frac{BLv_t}{R} \]

Now because there is a current in the loop, there is a magnetic force, \( \vec{F}_B \), acting on it:

\[ |\vec{F}_B| = \int \vec{\text{ind}} \times \vec{B} \, d\vec{A} = \int \text{ind} d\vec{x} \times \vec{B} \]

Forces cancel on vertical legs so

\[ |\vec{F}_B| = ILB \]

\[ = \frac{B^2 L^2 v_t}{R} \]

No acceleration means total force = 0, balanced forces, so

\[ |\vec{F}_{\text{gravity}}| = |\vec{F}_B| \]

\[ mg = \frac{B^2 L^2 v_t}{R} \]
So \[ V_L = \frac{mgR}{B^2L^2} \]

31-34) Make a function for \( B \):

\[ B = 29.6 + 0.2 \sin(2\pi(15)t) \]

Think of an imaginary loop at \( r = 1.6 \text{ cm} \). The flux through this loop is:

\[ \Phi_{1.6} = \oint B \, dl = \Phi_A = B \pi (1.6 \text{ cm})^2 \]

and so

\[ \frac{d\Phi}{dt} = -2\pi (15)(0.2) \int \cos(2\pi(15)t) \pi (1.6) \]

\[ = 1.5 \cos(2\pi(15)t) = 0.16 \]

So \[ \frac{d\Phi}{dt} \bigg|_{\text{max}} = 1.5 \]

By Faraday's law, eq. 31-22)

\[ \oint E \cdot dl = -\frac{d\Phi}{dt} \]
Along loop $|E|$ is constant as is $E \cdot ds$, by cylindrical symmetry so:

$$\int E \cdot ds = 2\pi r E = 2\pi E (0.165)$$

and solving $3122$ for $E$

$$E = \frac{261}{2\pi (0.165)} = \frac{1}{2\pi (0.165)}$$

$$\approx 7.5$$

(Warning: Final answer may be off due to calculation errors; however, this is correct way to do the problem)

31-38) Cross sectional view

Choose amperian vortex shaped loop as w/ regular (height = $Q$)
Solenoid: We are ONLY if $B = 0$.
outside and we assume that $B$ is constant inside as $\mu$ reg. solenoid.
So the only part of the loop that has nonzero $\vec{B} \cdot d\vec{l}$ is the inside vertical leg, and

$$\oint_{\text{box loop}} \vec{B} \cdot d\vec{l} = B l$$

And so by Ampere's law,

$$\oint_{\text{loop}} \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{loop}}$$

then

$$B l = \mu_0 I_{\text{enclosed in loop}}$$

The fraction of the total current $i$ enclosed in loop is $\frac{l}{W}$.

So $Bl = \mu_0 \frac{l}{W} i \Rightarrow \boxed{B = \frac{\mu_0 i}{W}}$
b) Use energy:

Total energy stored in fields \( E_F \) equals total energy in inductor \( E_L \).

\[ E_F = E_L \]

\( E_F = \text{Energy density} \times \text{Volume} = \int u_0 \, dV \)

Eq. 31-56 \( u_0 = \frac{B^2}{2\mu_0} \). Since \( B \) is constant:

\[ E_F = \int u_0 \, dV = \int \frac{B^2}{2\mu_0} \, dV = V_{\text{coil}} \frac{B^2}{2\mu_0} \]

\[ = \frac{\pi R^2 W B^2}{2\mu_0} \]

\[ = \frac{\pi R^2 W}{2\mu_0} \left( \frac{\mu_0 I}{W} \right)^2 = \frac{\pi R^2 W \mu_0 I^2}{2} \]
Now by 3151

\[ E_L = \frac{1}{2} Li^2 \]

So use the energy equation to solve for \( L \)

\[ E_L = E_F \]

\[ \frac{1}{2} Li^2 = \frac{\pi R^2 \mu_0 W}{2} \]

\[ L = \frac{\pi R^2 \mu_0 W}{2} \]

**NOTE:** Using energy equation \( E_F = E_L \) is often easier than using the definition of inductance. Use this method when \( B \) is constant over some volume.
This time, use def. of inductance:

\[ L = N \frac{\Phi}{i} = \frac{\Phi}{i} \]

Note that \( B \) is constant along vertical strips and that the \( B \)s from both wires at any distance \( x \) between add up. So the flux through one of these strips is:

\[ d\Phi = B \cdot dA = B \delta l \, dx = (B + B) \, l \, dx \]

And therefore:

\[ \Phi = \int_{a}^{b} \left( \frac{\mu_0 i}{2\pi x} + \frac{\mu_0 i}{2\pi (d-x)} \right) l \, dx \]

\[ = \frac{\mu_0}{2\pi} \left[ \ln x \bigg|_{a}^{d-a} - \ln (d-x) \bigg|_{a}^{d-a} \right] \]
\[
\frac{\mu_0 l i}{2\pi} \left( \ln \left( \frac{d-a}{a} \right) - \ln \left( \frac{a}{d-a} \right) \right) = \frac{\mu_0 l i}{2\pi} \left( \ln \left( \frac{d-a}{a} \right) + \ln \left( \frac{d-a}{a} \right) \right) = \frac{\mu_0 l i \ln \left( \frac{d-a}{a} \right)}{2\pi}
\]

Finally, by definition of inductance:

\[
L = \frac{\Phi}{i} = \frac{\mu_0 l i \ln \left( \frac{d-a}{a} \right)}{2\pi}
\]

31-55 (a)

Use Kirchhoff's law around loop with \( R + L \) (abcd), first applying current conservation:

\[
i_1 = i_2 + i_3 \Rightarrow i_3 = i_1 - i_2
\]

(if we had chosen opposite direction for \( i_3 \), this eq. would change)

yields the following differential equations...
\[-L \frac{di_2}{dt} + i_3 R = 0\]

rewritten in terms of \(i_2, i_1\)

\[-L \frac{di_2}{dt} + (i_1 - i_2) R = 0\]

or

\[\frac{di_2}{dt} + \frac{i_2}{L/R} = \frac{i_1 R}{L}\]

This equation has the exact same form as eq. 28-38 (i.e. you've seen it before) so, just following the derivation there, with the replacements: (see p. 650)

\[RC \rightarrow L/R \quad \text{and} \quad \frac{\varepsilon}{R} \rightarrow \frac{i_1 R}{L}\]

(we can do this because we are told that \(i_1\) is a constant.)

We can just write down the solution (like 28-38)

\[i_2 = i_1 \left(1 - e^{-\frac{t R}{L}}\right)\]
b) \[ i_3 = i_1 - i_2 \]

\[
\begin{align*}
i_1 (t = \frac{L}{R} \ln 2) &= i_1^0 + \frac{Q}{C} \\
i_2 (t = \frac{L}{R} \ln 2) &= i_1 (1 - e^{-\frac{L}{R} \frac{\ln 2}{L}}) = i_1 (1 - e^{-\frac{L}{R}}) \\
&= i_1 (1 - e^{-L}) = \frac{i_1}{2} = \frac{i_1}{2}
\end{align*}
\]

therefore

\[ i_3 = i_1 - \frac{i_1}{2} = \frac{i_1}{2} = i_2 \]

---

Use def of Mut. ind.

\[ M_{21} = \frac{N_2 \Phi_{21}}{i_1} \quad \text{or} \quad M_{12} = \frac{N_1 \Phi_{12}}{i_2} \]

For a small slice of the 2 solenoids:

\[ N_4 \quad \text{is easy; just} \ln \frac{d}{r} \]

\[ \Phi_{12} \quad \text{is just flux of} \ B_z \quad \text{through} \ i. \]
$$\Phi_{12} = \int B_2 \, dA = B_2 A_1$$

$$= B_2 \left( \pi R_1^2 \right) = \left( m_0 n_2 l_2 \right) \left( \pi R_1^2 \right)$$

($B_2$ from $\Phi_{\text{sol}}$ equation)

$$= m_0 n_2 \pi R_1^2 l_2$$

so

$$M_{12} = m_0 n_2 \pi R_1^2 l_2$$

The reason $M_{12} = M_{21}$ only depends on $R_1$ is because when you solve for $M_{12}$ this way only $A_1$ comes in, but even if you solve for $M_{21}$ the $B$-field of loop 1 only exists inside loop 1 so $\Phi_{21}$ would still be non-zero over $A_1$ only.

A more elegant way to explain it is: inside solenoid 1 is the only place the fields $B_1 + B_2$ "mix." Nothing is different than a regular $R_2$ solenoid outside this volume.
Remember \( M_{i2} = M_{2i} \) always.

Solve for \( M_{21} \). We need \( \Phi_{21} \). Consider one loop of outer coil 2:

We need flux of \( B \) over cross sect. area of 2. \((A_2)\)

However \( B \) is non-zero only inside \( A_1 \). So we only need flux through 1.

In essence \( \Phi_{21} = \Phi_{11} \). \( B \) is constant over vertical strips as drawn because \( B_{\text{toroid}} = \frac{M_0 N_i i x}{2 \pi r} \). So:

\[
\Phi_{21} = B_1 \cdot dA_1 = B_1 h dr = \frac{M_0 N_i i x h}{2 \pi r} dr
\]

\[
\Phi_{21} = \int_a^b \frac{M_0 N_1 N_2 i x h}{2 \pi r} dr = \frac{M_0 N_1 N_2 i x h}{2 \pi} \ln \frac{b}{a}
\]

So:

\[
M_{21} = N_2 \Phi_{21}
\]

\[
= \frac{M_1 N_1 N_2 i x h \ln b}{2 \pi} \frac{b}{a}
\]
a) Again, remember $M_{21} = M_{12}$ always.

Call loop inductor 2 and solve for $M_{21}$ then:

$$N_2 = N$$

For flux $\Phi_{21}$ use horizontal strips of width $dr$ as shown plus length $l$.

$$B_{1} = B_{wire} = \frac{\mu_0 i}{2\pi r} = constant$$

Along Strip

$$\Phi_{21} = B_{1} IA_{strip} = \frac{\mu_0 i l}{2\pi r}$$

$$\Phi_{21} = \int_{a}^{a+b} \frac{\mu_0 i l}{2\pi r} dr = \frac{\mu_0 i l \ln \left( \frac{a+b}{a} \right)}{2\pi}$$

So $M_{21} = M = \frac{N \mu_0 l \ln \left( \frac{a+b}{a} \right)}{2\pi}$

$$b) \left[ 0.3 \right] \left[ \left( \frac{1}{0.01} \right) \left( 4\pi \times 10^{-7} \right) \ln \left( \frac{0.01 + 0.08}{0.01} \right) \right] \frac{1}{2\pi} = \frac{1}{2\pi} 8.3 \times 10^{-5} \, H \leq 1.4 \times 10^{-5} \, H$$

*NOTE: Recheck math*