Problem 1. 25-8P

a) Use \( \Delta V_{ab} = - \int_{a}^{b} \mathbf{E} \cdot d\mathbf{s} \) and take radial path from \( a \) to radius \( r \). Then \( \mathbf{E} \cdot d\mathbf{s} = E \cdot dr \)

So \( \Delta V = V(r) - V(0) = - \int_{0}^{r} E(r) dr = - \int_{0}^{r} \frac{q}{4\pi\varepsilon_{0}R^{3}} r \ dr \)

Since \( V(0) = 0 \), \( V(r) = - \frac{q}{8\pi\varepsilon_{0}R^{3}} r^{2} \)

b) \( V(R) - V(0) = - \frac{q}{8\pi\varepsilon_{0}R} \)

c) The center of the sphere is at the highest potential.

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Note: Let's compare above result with what we would get for \( V(r=\infty) = 0 \). For \( r > R \), \( E \) looks like that of point charge. So \( V(r) = \frac{q}{4\pi\varepsilon_{0}r} \Rightarrow V(R) = \frac{q}{4\pi\varepsilon_{0}R} \)

Now, for \( R < r \), \( V(r) = V(R) - \int_{R}^{r} \frac{q}{4\pi\varepsilon_{0}r^{3}} r dr \)

So \( V(r) = \frac{q}{4\pi\varepsilon_{0}R} - \frac{1}{4\pi\varepsilon_{0}R^{3}} \left( \frac{r^{2}}{2} \right) \bigg|_{R}^{r} = \frac{q}{4\pi\varepsilon_{0}R} \left( 1 - \left( \frac{r^{2}}{2R^{2}} - \frac{1}{2} \right) \right) \)
or \( V(r) = \frac{q}{4\pi\varepsilon_{0}R} \left( \frac{3}{2} - \frac{r^{2}}{2R^{2}} \right) = \frac{q}{8\pi\varepsilon_{0}R} (3 - \frac{r^{2}}{R}) \)

Then \( V(0) = \frac{3}{8} \frac{q}{\pi\varepsilon_{0}R} \)

The difference in 1st & 2nd values for \( V(r) \) just due to different \( V(0) \)
Problem 2 (cont)

With the Dipole oriented as shown, by symmetry \( E_y = 0 \) everywhere along the \( y-z \) plane through \( x=0 \).
\( \vec{E}(x=0, y, z) \) points only in \( \hat{r} \) direction.

a) Now suppose I bring a charge in from \( \infty \) along this plane. If I follow a path in the plane \( \oint \vec{E} \cdot d\vec{s} = 0 \) because \( \vec{E} \perp \) plane so everywhere \( \vec{E} \cdot d\vec{s} = 0 \). Thus, all points on the plane are at the same potential, \( V=0 \) (because we have defined \( V=0 \) at infinity).

The plane includes \((0,0,0)\) so \( V(0,0,0) = 0 \).

b) In class I showed that for multiple charges
\[
V = \frac{\sum q_i}{\epsilon_0 r_i}
\]

So, on the x axis
\[
V(x) = \frac{kq}{|x-q_2|} + \frac{k(-q)}{|x+q_2|}
\]

Note: We need to take absolute value of \( x \pm q/2 \) because distances (as opposed to displacements) are positive always
or
\[
V(x) = kq \left( \frac{1}{|x-q_2|} - \frac{1}{|x+q_2|} \right) \] which \( = 0 \) at \( x=0 \) as expected
Problem 2 (cont.)

c) Now take limit as \( x \to \infty \). You cannot say \( \frac{1}{|x \pm \sqrt{2}|} \to \frac{1}{x} \).

You need to use binomial approximation, \( (1+\delta)^n \approx 1+n\delta \).

Then, \( \frac{1}{|x \pm \sqrt{2}|} = \frac{1}{x} \frac{1}{|1 \pm \sqrt{2}|} \) for \( x > 0 \).

Since we're taking \( x \to \infty \) limit, we can forget about \( 1 \) and

\[
\frac{1}{|x \pm \sqrt{2}|} \to \frac{1}{x} (1 \pm \sqrt{2})^{-1} = \frac{1}{x} \left( 1 \mp \frac{\sqrt{2}}{2} \right)
\]

So as \( x \to \infty \) \( V(x) \to \frac{kq}{x} \left( (1 + \frac{\sqrt{2}}{2}) - (1 - \frac{\sqrt{2}}{2}) \right) = \frac{kq}{x^2} \)

Now, the large-distance form for the \( \vec{E} \) field of the dipole is \( \vec{E} = \frac{2kq\vec{a}}{4\pi x^3} \) for \( x > 0 \).

Since \( V = 0 \) at \( x = \infty \), \( V(x) = -\int_{\infty}^{x} \frac{2kq\vec{a}}{4\pi x^3} \, dx \)

\[ V(x) = \frac{2kq\vec{a}}{2x^2} \left| _{\infty}^{x} \right. = \frac{kq}{x^2} \]

So, our explicitly calculated \( V \) agrees with result from integration of dipole (large-distance) \( \vec{E} \) field.

Note: Since \( V(0) = 0 \), you might think you could do the dipole field integral from \( x = 0 \) as well as from \( x = \infty \). But \( \vec{E} = \frac{2kq\vec{a}}{4\pi x^3} \) is only valid for large \( x \) so you have to start at \( x = \infty \).
To obtain the total electric potential on the z axis we consider the contribution from segment $\Delta \theta$ of the arc. This segment contains charge $\lambda R \Delta \theta$.

So \[ dV = \frac{1}{4\pi \varepsilon_0} \frac{\lambda R \Delta \theta}{r(\theta)} \] with \( r(\theta) = \sqrt{z^2 + R^2} \)

Since the right-hand side has no explicit dependence on $\theta$, we can trivially integrate:

\[
\int dV = \frac{1}{4\pi \varepsilon_0} \frac{1}{\sqrt{z^2 + R^2}} \lambda R \int d\theta \quad \text{Note:} \quad \lambda = \frac{q}{2\pi R} = 2\pi
\]

So \[ V = \frac{1}{4\pi \varepsilon_0} \frac{q}{\sqrt{z^2 + R^2}} \quad \text{on z axis} \]

b) Now, the z component of $\vec{E}$ is given by,

\[
E_z = -\frac{dV}{dz} = \frac{-1}{4\pi \varepsilon_0} \frac{q}{(\sqrt{z^2 + R^2})^3} \left(-\frac{1}{2}\right) \times 2z
\]

\[ = \frac{1}{4\pi \varepsilon_0} \frac{9z}{(z^2 + R^2)^{3/2}} \]

This agrees with the explicit calculation of $E$ in Section 23-6 of your book.
Problem 4, 25-52

If the two spheres are far apart, they have no effect on each other so I can treat sphere 1 as a uniformly charged conducting sphere with \( V_1(R) = \frac{1}{4\pi\varepsilon_0} \frac{q_1}{R_1} \)

I can treat sphere 2 as initially uncharged with potential at its surface = 0.

a) Once the spheres are connected by a conducting wire, the potentials must be the same \( V_1 = V_2 \) because there is no \( E \) field in the conductor.

b) Once sphere two becomes charged, \( V_2 = \frac{1}{4\pi\varepsilon_0} \frac{q_2}{R_2} \)

Then, we require \( V_1(R_1) = V_2(R_2) \Rightarrow \frac{q_1}{R_1} = \frac{q_2}{R_2} \)

but, we also require \( q_1 + q_2 = q \)

So \( q_1 + \left(\frac{R_2}{R_1}\right)q_1 = q \Rightarrow q_1 = \frac{1}{1 + \frac{R_2}{R_1}} q = \frac{R_1}{R_1 + R_2} q \)

Then \( q_2 = q - q_1 = \left(\frac{R_2}{R_1 + R_2}\right) q \)

\( q_1 = \frac{1}{3} q, \; q_2 = \frac{2}{3} q \)

C) \( \sigma_1 = \frac{q_1}{4\pi R_1^2} \; \sigma_2 = \frac{q_2}{4\pi R_2^2} \)

So \( \sigma_1 = \frac{1}{4\pi R_1} \frac{q}{R_1 + R_2} \; \sigma_2 = \frac{1}{4\pi R_2} \frac{q}{R_1 + R_2} \)

So \( \sigma_1/\sigma_2 = \frac{R_2}{R_1} = 2 \)
Problem 5

If we have a charge \( q \) on the outer surface of inner sphere, we have charge \(-q\) on inner surface of outer sphere.

Without going through explicit calculation, I'll assume you know that between the spheres \( E(r) = \frac{1}{4\pi \varepsilon_0} \frac{q}{r^2} \) \( \hat{r} \).

a) Then \( \Delta V \) between spheres is just \(-\int_a^b E(r) dr\). Since I am free to choose the path 2 I choose a radial path (inner to outer) where \( \vec{E} \parallel d\vec{s} \).

So
\[
\Delta V = \frac{-q}{4\pi \varepsilon_0} \int_a^b \frac{1}{r^2} dr = \frac{-q}{4\pi \varepsilon_0} \left( \frac{1}{a} - \frac{1}{b} \right)
\]

Rewrite, \( \Delta V = \frac{-q}{4\pi \varepsilon_0} \left( \frac{b-a}{ba} \right) \).

To calculate \( C \), we take \( |\Delta V| \), \( C = \frac{q}{|\Delta V|} \).

So,
\[
C = 4\pi \varepsilon_0 \frac{ba}{b-a}
\]

b) For \( b \gg a \), \( \frac{ba}{b-a} \rightarrow a \) so \( C \rightarrow 4\pi \varepsilon_0 a \) as calculated in book for an isolated sphere.

c) For \( b-a \equiv \delta \ll a \), \( C \rightarrow \frac{\varepsilon_0}{\delta} \left( 4\pi a^2 \right) = \frac{\varepsilon_0 A}{\delta} \)
Problem 6, 26-15

We can treat this problem as two planer capacitors in series:

\[ C_1 = \frac{A \varepsilon_0}{d_1}, \quad C_2 = \frac{A \varepsilon_0}{d_2} \]

\[ C_{eq} = \left( \frac{1}{C_1} + \frac{1}{C_2} \right)^{-1} = \left( \frac{C_2 + C_1}{C_1 C_2} \right)^{-1} = \frac{C_1 C_2}{C_1 + C_2} \]

So \[ C_{eq} = \frac{A^2 \varepsilon_0^2}{d_1 d_2} = \frac{A \varepsilon_0}{d_1 + d_2} \]

Now, the sum of the separations between the plates, \( d_1 + d_2 = a - b \) which is constant regardless of the position of the center section giving

\[ C_{eq} = \frac{A \varepsilon_0}{a - b} \]

Problem 7, 26-33

The potential energy of a charged capacitor is given by \( U = \frac{1}{2} CV^2 \) with \( V = \frac{q}{C} \Rightarrow U = \frac{1}{2} \frac{q^2}{C} \)

For parallel-plate capacitor, \( C = A \varepsilon_0 / d \)

So \[ U = \frac{1}{2} \frac{q^2 d}{A \varepsilon_0} \]

So, since \( U \) depends linearly on the separation

\[ dU \Delta x = U(x+\Delta x) - U(x) = \frac{1}{2} \frac{q^2}{A \varepsilon_0} (x+dx-x) = \frac{1}{2} \frac{q^2}{A \varepsilon_0} dx \]
Problem 7 (cont.)

Now, last semester you saw that we could obtain force from the potential energy function, \( F_x = \frac{dU}{dx} \)

So, we obtain \( F = \frac{1}{2} \frac{q^2}{\epsilon_0} \)

Aside:

If you didn’t follow the books directions you might have argued the following:

\( E \) at Conductor Surface is \( 1E_b = \frac{q}{\epsilon_0} = \frac{q}{\epsilon_0} \)

(due to other plate)

There’s charge \( q \) on the conductor, \( F = qE = \frac{q^2}{\epsilon_0} \)

Why is this wrong -- i.e. where does the \( \frac{1}{2} \) come from?

The answer is that the charges on one plate feeling the force due to charges on the other plate also modify the \( E \) field. Suppose there’s a finite width to distribution of charges that are feeling the force. Now we have calculated that \( E \) varies \( E = \frac{q}{\epsilon_0} \) linearly in a uniformly distributed slab of charge. In this case from \( \frac{q}{\epsilon_0} \) to 0.
Problem 7, cont.

Aside continued:

Since the field decreases linearly, the average $E$ is $\frac{1}{2} \frac{q}{\varepsilon_0 A}$.
So the force averaged (actually -- summed) over all charges is $\frac{1}{2} \frac{q^2}{\varepsilon_0 A}$.

Thus, $\frac{1}{2}$ results from the fact that the charges feeling the $E$ field also cause $E$ to decrease to zero.

This result is very general, the force per unit area at conductor surface is
\[
\frac{E}{A} = \frac{1}{2} \frac{q^2}{A^2 \varepsilon_0} = \frac{1}{2} \frac{\sigma^2}{\varepsilon_0}
\]
because the $E$ field is directly determined by $\sigma$.

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b) See last part of aside, $E = \frac{\sigma}{\varepsilon_0}$

So $F/A = \frac{1}{2} \varepsilon_0 E^2$ -- generally.

Problem 8, 26-40

We can solve this problem using $E$ fields instead of using formula for Capacitors in parallel. The two plates are each @ same potential across their surface so the potential difference between the plates is the same on both halves.

\[
\frac{\frac{1}{2}A}{\frac{1}{8}A} \quad E_1, \downarrow \uparrow K_1 \downarrow \uparrow \quad \cdots \quad E_2, \downarrow \uparrow K_2 \downarrow \uparrow
\]

$E$ must be the same in both regions, $E_1 = E_2$.\} d
Problem 8 (cont.)

Now, if we apply generalized Gauss' law at surface of one plate:

\[ \mathcal{E}_0 \oint \vec{E} \cdot d\vec{A} = q_{enc} \]

Then due to planar symmetry this reduces to

\[ \mathcal{E}_0 KEA = \sigma A \Rightarrow E = \frac{\sigma}{K\mathcal{E}_0} \]

So in Region 1, \( \sigma_1 = K_1 \mathcal{E}_0 E = K_1 \mathcal{E}_0 \frac{\Delta V}{d} \)

" " 2, \( \sigma_2 = K_2 \mathcal{E}_0 \frac{\Delta V}{d} \)

Then the total stored charge is \( q = \frac{A}{2} \sigma_1 + \frac{A}{2} \sigma_2 \)

So \( q = \frac{1}{2} A \left( K_1 + K_2 \right) \mathcal{E}_0 \frac{\Delta V}{d} \Rightarrow C = \frac{A \mathcal{E}_0}{d} \left( \frac{K_1 + K_2}{2} \right) \)

This result is exactly what you would get if you solved the problem as two capacitors in parallel because that's exactly what the two \( \frac{1}{2} \)'s are. They are in parallel because they have the same potential difference across them.

Problem 9, 26-45

If we assume a charge (free) \( q \) on inner shell, then we can apply generalized form of Gauss' law to determine \( E \).
Problem 9 (cont.)

\[ \oint \mathbf{KE} \cdot d\mathbf{A} = q_{\text{enc}} \]

Take spherical Gaussian surface of radius \( r \).

Then, \[ \oint \mathbf{KE} \cdot d\mathbf{A} \Rightarrow \varepsilon_0 \varepsilon_k 4\pi r^2 E \]

So \[ E(r) = \frac{q}{4\pi \varepsilon_0 k r^2} \] for \( a < r < b \).

Since this looks like \( E \) of point charge, \( V(r) = \frac{q}{4\pi \varepsilon_0 k r} \)

and \[ V(a) - V(b) = \frac{q}{4\pi \varepsilon_0 k} (\frac{1}{a} - \frac{1}{b}) = \frac{q}{C} \]

a) So \( C = 4\pi \varepsilon_0 k \times \frac{ab}{b-a} \Rightarrow \) similar to result from problem 5 but with factor \( k \).

b) So if we know \( V(a) - V(b) = V \), \( q = V 4\pi \varepsilon_0 k \frac{ab}{b-a} \)

c) Look more closely at surface of inner shell assuming \( q \) is positive: The charges are the induced at inner surface of dielectric by the \( E \) field.

Due to spherical symmetry, the induced charge has uniform density \( \sigma^- \) with \( \sigma^- \) a positive quantity.
Problem 9, (cont.)

If we apply the original form of Gauss' law,

\[
\mathbf{E}_0 \oint \mathbf{E} \cdot d\mathbf{A} = q_{\text{enc}} \quad \text{with Gaussian surface a sphere of radius } r
\]

the \( \mathbf{E}_0 4\pi r^2 E = q - 4\pi a^2 \sigma^- = \frac{q}{K} \) plugging in \( E \) from above.

Since the area of the inner surface of the dielectric is constant we can write \( 4\pi a^2 \sigma^- = q' \)

So \( q - q' = \frac{q}{K} \) or \( q' = q \left(1 - \frac{q}{K}\right) \)

Note: I explicitly took the sign out of \( \sigma^- \) and put it in but for \( q' \) we should put it back in,

i.e. \( q' = -q \left(1 - \frac{q}{K}\right) \)