Applications of Probability Theory

to Nuclear Particle Detection

(Part 1)

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The random nature of the fundamental processes involved in the emission and detection of nuclear particles leads to many important differences between such measurements and most ordinary physical measurements. Since it has been the experience of the authors that a proper appreciation of these factors requires good understanding of the basic principles involved, emphasis has been placed on the derivation of all relations from elementary considerations.

The following problems will be considered: (a) If the "mean" expected value for a count is \( \mu \), what is the probability \( P_n(\mu) \) that exactly \( n \) counts will be obtained? (b) If \( n \) counts are actually obtained, what is the probability that the "mean" expected value lies between \( u \) and \( (u + du) \)? (c) If a counter, counter circuit, or mechanical register has a "dead" (insensitive) time \( r \) after each pulse, what fraction of the (random) input pulses will not be detected due to this effect (counting losses)? (d) What is the theory of the regularizing action of a scale-of-\( m \) scaling circuit in reducing counting losses by a mechanical register?

Before these main problems are discussed, it will be useful first to consider a few elementary problems, the solutions of which will facilitate the derivation of the basic relations which will be needed later. For this purpose, the following simple problems, involving successive throws of a single die, and other problems will be used.

**Introductory Problems**

1. What is the probability that a given result will be obtained on a single throw of a die? **Soln.** Since all numbers from 1 to 6 are equally probable, the apriori probability is \( p = \frac{1}{6} \).

2. What is the probability of not throwing a given number on one throw of a die? **Soln.** Since there is unit probability that some result will be obtained, the probability is 1 minus the probability, \( \frac{1}{6} \), that the given number will be thrown. Thus, \( p = 1 - \frac{1}{6} = \frac{5}{6} \).

3. What is the probability that the given sequence, \([3, 4, 2, 6, 5]\), will be obtained on five successive throws? Here 4 and 6 mean that the 4 or the 6 should not be thrown on the given throw. **Soln.** In order that the complete sequence be obtained, it is necessary for each successive step to be satisfied. The five steps have the separate probabilities \( \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6} \). The probability is \( \frac{1}{6} \) that the first throw will be successful. If it is successful, then the chance is \( \frac{1}{6} \) that the next step will be successful, etc. The total probability for the complete sequence is thus the product of the separate probabilities of the individual steps and \( p = \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} = \left(\frac{1}{6}\right)^5 \).

4. What is the probability of throwing exactly one 6 in five throws? **Soln.** Any successful sequence of throws must have one throw giving a 6 and four throws not giving a 6. From problem 3, the probability \( p_1 \) of such a sequence is \( p_1 = \left(\frac{1}{6}\right)^5 \). The total probability is the sum of the separate probabilities of all possible sequences that will lead to the correct result. Since all successful sequences have the same probability \( p_1 \), the total probability is just \( p_1 \) times the number of permitted sequences. Since, in this case, a given successful sequence can be expressed in terms of the number of the throw on...
A thorough elementary development and discussion of the theory of statistical accuracy, counting losses, and scaling circuit regularizing action. This is the second of a series of articles devoted to the principles underlying the design and use of radiation measuring and detection instruments

which the 6 occurs, there will be just five possible places for the six to occur and $P = 5p^6(5\frac{4}{6})^6$. 

6. How many different groups of $n$ different numbers can be chosen from the numbers 1 to $N$? Two "different" groups must have at least one of their numbers different. (The same $n$ numbers arranged differently do not constitute a new group.) Soln. The first number selected may be any of the $N$ numbers so there are $N$ ways of choosing it. After the first number has been selected, there are $(N - 1)$ numbers from which the second number may be chosen. After the first and second numbers have been chosen, there are $(N - 2)$ numbers from which the third may be chosen. Thus, there are $N(N - 1)(N - 2)$ ways in which $n$ numbers may be selected from $N$ numbers. This, however, includes not only all different groups of $n$ numbers, but also all arrangements of the chosen numbers. To obtain the number of different groups, we must divide the above result by a factor equal to the number of ways $n$ numbers may be arranged. Thus, $(1,2,3), (3,2,1), (2,1,3), (2,3,1)$ and $(3,2,1)$ are the six possible arrangements for the three numbers indicated. With $n$ numbers, there are $n$ choices for first position, $(n - 1)$ choices for second position after the first position is selected, etc. This gives the "arrangement factor" $= n!$. The required result is

\[
\frac{N!}{n!(N - n)!} = \text{number of groups (1)}
\]

6. What is the probability of throwing exactly $n$ sixes in $N$ successive throws of a die? Soln. A given successful sequence requires $n$ throws giving 6 and $(N - n)$ throws not giving 6 and, from problem 3, has the probability $p = \left(\frac{5}{6}\right)^n\left(\frac{1}{6}\right)^{N-n}$. The number of possible successful sequences is the number of ways the $n$ throws for 6 can be selected from the $N$ total throws. From problem 5, this is just $\frac{N!}{n!(N - n)!}$ giving

\[
P = \left(\frac{1}{6}\right)^n \left(\frac{5}{6}\right)^{N-n} \frac{N!}{n!(N - n)!} \quad (2)
\]

The above examples have developed the necessary relations for the solution of the counting problem which will first be stated in a restricted fashion for simplicity and later in a more general fashion.

**Restriction Statement of Problem**

Assume that in a counting device for the detection of radioactive disintegrations there are exactly $N$ radioactive atoms present in a sample at the start of the counting period and that each has the same probability $p$ of disintegrating and producing a count during the counting interval. Then what is the probability $P_n$ that exactly $n$ counts will be obtained during the counting period? (Note that $p$ is the product of the separate probabilities that a given atom will disintegrate, times the probability that the given disintegration will be detected in the given time interval. In general, either or both of these separate factors may be very small so that $p$ is usually very small.)

**Solution.** This problem is exactly similar to problem 6 above. Here the tabulation of whether each atom will or will not give a count corresponds to whether each throw of the die will or will not give a 6. The term $p$ replaces (\( \frac{5}{6} \)) and (1 - $p$) replaces (\( \frac{1}{6} \)).

\[
P_n = p^n(1 - p)^{N-n} \frac{N!}{n!(N - n)!} \quad (3)
\]

This problem is known as Bernoulli's problem having equation 3 as the solution. It is of interest to consider some of the properties of the solution.

Since some result must be obtained

\[
\sum_{0}^{N} P_n = [p + (1 - p)]^N = 1 \quad (4)
\]

The term $P_n$ is the $(n + 1)$th term in the above-indicated binomial expansion. The mean value $u = \sum P_n$ is

\[
\sum_{0}^{N} P_n = \sum_{0}^{N} n P_n = \sum_{0}^{N} n P_n = \frac{Np}{M} \quad (5)
\]

where $M = N - 1$ and $m = (n - 1)$ and equation 4 is used. Thus

\[
u = \sum P_n = \sum_{0}^{N} n P_n = \frac{Np}{M} \quad (5)
\]

Similarly $n_u^2$ is of interest,

\[
n_u^2 = \sum_{0}^{N} n^2 P_n = \sum_{0}^{N} n^2 P_n = \frac{N(N - 1)p^2}{M(M - 1)} \quad (6)
\]

or

\[
n_u^2 = Np(1 + (1 - p)) \quad (6)
\]

where $\sum P_n$ refers to a different total $M$ rather than $N$.

The quantity $u = \sum P_n$ is the mean value for the number of counts expected from the measurement. It is also of interest to determine the amount by which any given measurement may be expected to deviate from the mean value. The quantity $\sigma$ used to denote this is called the "standard deviation" and $\sigma^2$ the "dispersion" of the distribution. $\sigma$ represents the "root mean square" deviation.

\[
\sigma^2 = \sum_{0}^{N} (n - u)^2 P_n = u^2 \sum_{0}^{N} P_n = \frac{Np}{M} \quad (7)
\]

The standard deviation is equal to $u_N^{1/2}$. Thus if 100 counts are expected on the average, the actual value of a given count has a good chance of differing from this by about $\pm 10$ or $10\%$, whereas for 10 counts the deviation is $\approx 0.1\%$. If $p$ is near unity, $\sigma \ll u_N^{1/2}$. This will happen for counting only when a short-lived radioactive material is counted over several half-lives and with high ($\approx$ unity) detection efficiency. The significance of the decreased uncertainty from $u_N^{1/2}$ that now most of the active atoms originally present are counted and are not just a small sample. In the limit where all of the atoms are counted, there is no uncertainty as to how many were originally present. This situation is unusual since $p$ is usually extremely small because of a combination of the factors involved in its definition.

**Poisson Distribution**

When $p$ is very small and $N$ is very large and $n \ll N$, equation 3 becomes simplified by the substitutions $\sum_{0}^{N} (n - u)^2 P_n = \sum_{0}^{N} P_n = \frac{Np}{M}$ and $p^2N = uN$ giving

\[
P_n = u^n e^{-u}/n! \quad (9)
\]

which is known as the "Poisson" distribution. This distribution is much more general than equation 3 since, as is shown below, it may be derived with-
The General Statement of Problem

The derivation of equation 9 may be made very general using the law of large numbers. The original statement of the counting problem required a constant probability p for all atoms. In general, there is a certain probability (which is usually zero) that any given atom in the universe will disintegrate and be detected by producing a count in the counter during the measuring interval. The general problem may thus be expressed as below.

If \( p_i \) represents the probability that the \( i \)th atom in the universe will produce a count during the measuring interval, what is the probability \( P_e \) that exactly \( n \) counts will be obtained?

Assume that \( p_i \ll 1 \) and \( np_i \ll 1 \) in general and let

\[
u = \sum_i p_i \tag{10}\]

The derivation of the general term \( P_e \) may be understood more readily by considering \( P_e \) for \( n \) atoms, etc. First, the probability that no counts are obtained is the product of the separate probabilities (1 - \( p_i \)) that each atom shall give no count. Thus

\[
P_0 = \phi e^{v} = e^{-\sum p_i} = e^{-\nu} \tag{11}\]

The significance of the separate terms in equation 9 are now apparent. \( e^{-\nu} \) is the probability that all other atoms do not give a count. \( \nu \) is the "arrangement" factor as discussed in problem 5 above. \( P_e \) represents the product of the separate \( \sum p_i \) terms for each of the \( n \) chosen atoms. Actually, equation 11 should omit the terms in each sum corresponding to the particular atoms selected by the preceding sum terms (see problem 5). For \( np_i \ll 1 \) or \( np_i \ll \nu \), this neglects relatively few terms in the infinite sums to give negligible error, and it greatly simplifies the expression of equation 11.

Properties of Poisson Distribution

Since the Poisson distribution (equation 9) has been shown to apply quite generally to counting problems involving small sampling (i.e., \( p_i \ll 1 \) and \( np_i \ll 1 \)), it is of interest to derive some of the properties of this distribution similar to those of equations 4-8 for equation 3.

\[
\sum_i P_i = e^{-\nu} \sum_i \frac{n!}{i!} \frac{\nu^i}{i!} = 1 \tag{12}\]

or

\[
\nu_{av} = \nu \tag{13}\]

as expected.

\[
\nu_{av}^2 = \sum_i n^2 P_i = e^{-\nu} \sum_i \frac{n^2}{(n-1)!} \tag{14}\]

or

\[
\nu_{av}^2 = \nu^2 + \nu \tag{14}\]

Using equation 7, we obtain \( \sigma^2 = \nu \) or \( \sigma = \nu^{1/2} \) as for the Bernoulli distribution when \( p \ll 1 \).

Inverse Problem for Counting

The result of the direct problem, as given by equation 9, may be written

\[
P_e(\nu) = e^{-\nu}/\nu! \tag{15}\]

showing that it is a function both of \( \nu \), the number of counts actually received, and of \( \nu = \sum_i p_i = \nu_{av} \), the mean-expected-number of counts. In actual practice, when counting, the directly measured quantity is \( \nu \) and the desired quantity is \( \nu_{av} \). The inverse problem may be stated thus: For an actual count of \( \nu_{av} \), what is the probability that the actual value of \( \nu \) lies between \( u \) and \( (u + du) \)? The philosophical justification for such a question lies in the assumption that, in principle, the experiment can be repeated as many times as desired to obtain \( \nu \) with any desired accuracy by increasing the total volume of data. A little thought will show that equation 9 is also the solution of the inverse problem when written

\[
P_e(\nu, u) = \int_u^{(u + du)} e^{-\nu}/\nu! \, du \tag{16}\]

It should be noted that \( n \) is constant and \( u \) variable in equation 16 while \( u \) was fixed and \( n \) variable in equation 9. Also \( n \) is variable only in integer steps while \( u \) is continuously variable.

The properties of equation 16 may be examined as were equations 3 and 9.

\[
\frac{d}{du} \int_0^u e^{-\nu}/\nu! \, du = \frac{d}{du} \int_0^u \nu_{av}^2 \, du = \nu_{av} \tag{17}\]

\[
\nu_{av} = \int \nu e^{-\nu} \, d\nu \tag{18}\]

The result (that the mean expected value \( \nu_{av} \) for \( u \) is larger than \( n \)) is unexpected at first but may be easily understood from a simple example. Thus, if \( n = 0 \), it is clear that \( u \) is not necessary zero; therefore \( \nu_{av} \) cannot be zero. Note, by contrast, that if \( \nu_{av} = 0 \), then \( n = 0 \) is the only possible result.

The most probable value \( \nu_{av} \) of \( u \) is the maximum of equation 16.
It will also be shown in a future article that the Geiger counter counting loss is actually a complex function of the "input" counting rate even when fast circuits are used. In practice, the form of the dependence may be sensitive to the actual circuit parameters. If the "regulation" of the high voltage and bias voltages as a function of counting rate is not good, then the circuit may count with relatively small losses up to a certain rate and block for higher rates. This type of behavior can be avoided, however, by proper circuit design, and a response similar to the idealized types discussed below can be obtained. Also when the theory of counting losses is developed, one of these idealized behaviors is assumed to apply.

Counting Loss Problem

The "input" counting rate \( N \) is the counting rate which would be obtained if the insensitive time were zero, and the "output" counting rate \( n \) is the rate when there is an insensitive \( t \) after each output pulse. In this statement of the problem, it is assumed that a new input count during the insensitive period, \( t \leq r \), after the previous output pulse, will not give an output pulse and also will not extend the insensitive time. A new input is always assumed. The problem is to find the relation between \( n \) and \( N \).

Solution. The counting loss will be caused by those input counts which arrive during the insensitive periods. Since each output count has an insensitive time \( t \) and there are \( n \) output counts per second, on the average, the total insensitive time per second is \( Nt \). The average number of input counts during the insensitive period is just \( Nn \), however, so the loss is \( N(1-n) - nNt \) per second. This has the solutions:

\[
N = n/(1 - n) \quad \text{and} \quad n = N/(1 + Nr) \quad (29)
\]

This shows that the percentage loss is proportional to the output rate. The maximum output rate as \( N \to \infty \) is just \( r \). For a dead time \( t = 10^{-3} \) sec, the output rate of 100 per sec gives a 10\% counting loss, and the rate must be held to 10 per sec to have only a 1\% counting loss. In practice, the counting rates used may be such that \( 1-2\% \) counting losses are obtained in order to obtain a large total count in a short time for good statistical accuracy. The counting loss is then corrected by the use of equation 29, when \( r \) has been previously measured.

A second common statement of the counting loss problem modifies the previous statement as follows. It is again assumed that a new input count arriving during the insensitive period will not be counted, but it is assumed that the insensitive time is extended by the amount \( r \) after the (undetected) count arrives. Again, the relation between \( N \) and \( n \) is desired.

Solution. This problem is solved most readily by considering the case of a very high input rate. The system will remain blocked if the input counts are always spaced by less than \( r \). In general, another output pulse is obtained once every time the system recovers from blocking. Hence, every time a timing spacing \( \geq r \) occurs between successive input counts. Thus, \( n = N \times (probability \ of \ zero \ counts \ during \ the \ time \ r \ after \ a \ given \ count) \). Since the mean number of input counts expected during the time \( r \) is \( Nt \), the probability that no counts be received in a given interval \( t \) is:

\[
P(Nt) = e^{-Nt} \quad \text{and} \quad n = Ne^{-Nt} \quad (30)
\]

For \( (Nt)^2 \ll 1 \), equations 29 and 30 both give approximately

\[
n = N/(1 - Nt) = N(1 - nr) \quad (31)
\]

For \( Nt \gg 1 \), the output blocks as expected from the derivation. The maximum of equation 30 occurs for \( Nt = 1 \) which gives \( n = (re^{-r})^{-1} \).

Equations 29 and 30 are extreme cases of similar processes. What often happens is something intermediate. Thus, the system may require a time \( t \) to recover enough to give another output pulse if there was a large spacing between the two preceding input pulses. However, the extension of the insensitive time \( r \) after a given input pulse may be less than \( r \) for shorter spacings of the two preceding input pulses.

The best way to solve the problem for a given counting system is to vary the input rate experimentally in small steps over a wide range of known (random) input rates and measure the output rate to obtain directly the shape of the output vs input rate curve. To do this accurately is a very tedious process, however, and the shape of the curve may vary somewhat with time. The usual process, therefore, is to assume that either equation 29 or 30 applies and restrict measurements to the region where they are both closely given by equation 31. This requires that \( (Nt)^2 \ll 1 \).

Techniques for the experimental measurement of dead times will be discussed in greater detail in a forthcoming article of this series.

Scaling Circuit Regularizing Action

When the output from a system counting random pulses directly activates a mechanical register, a rate of only one count per sec will give a pulse out of the register if a count occurs at the rate. Since this high loss at moderate counting rates usually cannot be tolerated, it is customary to use a scaling circuit between the amplifier and the register circuits. A scale-of-\( m \) circuit is one which gives an output count for every \( m \) input counts. At present, the most common scaling circuit is derived from the basic scale-of-2 circuit giving scales of 2, 4, 8, 16, 32, 64, 128, etc. More recently, decimal type scales of 10, 100, etc., have been developed and may be used more frequently in the future. The relative advantages of each type will be discussed in a later section.

One obvious characteristic of a scale-of-\( m \) circuit is an average output rate lower than the input rate by the factor \( m \). Another less generally understood action is the "shortening" effect of the scaling circuit on the output. Thus, when random pulses are sent into a scale-of-128 circuit, the output pulses are quite uniformly spaced with only a small fractional variation in the spacings between successive output pulses. For a mechanical register with a resolving time of \( \frac{1}{2}t_0 \) sec, there will be a loss of less than 0.1\% of the register counts when a scale-of-32 circuit is used with an output rate of 10 counts per sec whereas 5\% of the counts would be lost for an output rate of only 1 count per sec without a scaling circuit. The use of a large scaling ratio thus decreases the output rate and also permits the average spacing of the output pulses to be only a little larger than the resolving time to give negligible counting losses. The theory of the scaling circuit regularizing action will be developed below.

A scale-of-\( m \) circuit is assumed to have a random input rate of \( N \) per sec corresponding to an output rate of \( N/m \) per sec. The resolving time of the scaling circuit is assumed to be negligible so counting losses other than those of the register are neglected here or are considered separately later. Thus, what is the probability, \( Q_m(z) \), that the actual timing spacing between output pulses lies between \( z \) and \( z + dz \) where \( z = Nt \) is the time in units of the average spacing between output pulses? And what is the probability \( Q_m(z) \) that the next output pulse will have arrived by \( t = z/N \)?

Solution. The circuit at the time of the next output pulse will have \( N \) input counts where \( n = 0, 1, 2, 3, \ldots \). The probability that there is exactly a given number \( n \) may be expressed by equation 9. The mean number of input counts expected during the interval \( t = z/N \) is just \( N \). The next output pulse will have been obtained if \( r \geq m \). The probability \( Q_m(z) \) that this will have happened is just the total probability that \( n \geq m \). From equation 9 this is

\[
Q_m(z) = \sum_{n=m}^{\infty} P_n(z) = [1 - \sum_{n=0}^{m-1} P_n(z)] dz \quad (32)
\]

where \( P_n(z) = (z/e^{-z})^n/n! \). The probability that the next output pulse
\[ 0 = \frac{d}{du} P_x(u) \quad \text{gives} \quad u_x = n (19) \]

This result shows that the most probable value of \( u \) is the measured value \( n \). Similarly

\[ u_x u_x^* = \int_0^{u_x} u_x P_x(u_x) du_x = \frac{(n + 2)!}{n!} \]

or

\[ u_x = u_x \sqrt{u_x + 1} \]

and

\[ \sigma^2 = u_x \quad \text{or} \quad \sigma = u_x^{1/2} \]

\[ = \frac{(n + 1)!}{(n + 1)!} \]

This result is similar to equation 15 except that \((n + 1)\) is used instead of \( u \). If \( n \gg 1 \), the distinction between \( n \) and \((n + 1)\) is not important.

**Gaussian Distribution**

Information concerning the shape of equation 9 is obtained by examining the ratio of the nth term to the preceding term

\[ P_n = \frac{u}{n} \]

\[ \text{or} \]

\[ = (2u)^{-1/2} e^{-u^2/2} \]

This shows that \( P_x \) increases with \( n \) for \( n < u \) and decreases with \( n \) for \( n > u \). For \( n >> u \), the terms decrease very rapidly with \( n \) since each term is only a small fraction of the preceding one. Similarly, for \( n << u \), the terms rapidly decrease in size as \( n \) is decreased.

From equation 22, the maximum occurs for \( n = u \); therefore \( P_x = P_{u-1} \) are the maximum values. The function \( P_x \) thus has a flat maximum near \( P_x \) and drops off rapidly on either side. Most of the area of a plot of \( P_x \) vs \( n \) occurs between \((u - \sigma)\) and \((u + \sigma)\) where

\[ \sigma = u^{1/2} \]

from equation 15. The value of \( P_x \) is obtained in a convenient form for the following formula by use of Stirling's formula

\[ n! \approx (2\pi n)^{1/2} (n/e)^n \]

\[ \text{giving} \]

\[ P_x = \frac{u e^{-x^2/2}}{n!} \]

\[ \text{or} \]

\[ = \frac{(n + y)^n}{n} e^{-y^2/2} \]

\[ \ln \left( \frac{n + y}{n} \right) = \ln \left( 1 + \frac{y}{n} \right) \]

\[ = \ln \left( 1 - \frac{1}{2} \left( \frac{y}{n} \right)^2 + \cdots \right) = y - \frac{y^2}{2n} \]

So

\[ f_x(y) dy = (2n)^{1/2} e^{-y^2/2n} \]

\[ = \frac{(2n)^{1/2} e^{-y^2/2n}}{2n} \]

which corresponds to equation 28.

The behavior of \( P_x(u) \) is not symmetric in \( n \) and \( u \) except to the degree that both can be approximately expressed as Gaussian distributions for \( u \) and \( n \) large and \( |u - n| \ll u \) and \( u - n \). In Fig. 1, \( P_x(u) \) has been plotted as a function of \( u \) for \( u = 0, 1, 2, 4, 8, 16 \) and as a function of \( u = 0.1, 1, 2, 4, 8, 16 \).

A logarithmic scale is used for the ordinate which should give the curves a parabolic shape if the Gaussian distribution were to apply rigorously. In Fig. 1, the Gaussian distribution, shown in equation 27, is plotted in a similar fashion (curve A).

**Finite Resolving Time**

All apparatus for the detection of individual particles have some resolving time such that particles which are spaced too closely will not give separate counts. This insensitive period, which occurs after the detection of one particle and before the system can respond properly to the next particle, is due to either the detection device itself (Geiger counter, proportional counter, ionization chamber, etc.) or to the associated amplifier circuit, level-setter circuit, ratemeter circuit, or a mechanical register. When accurate information concerning input rates is desired, it is necessary to correct for the counts lost in this fashion.

The following discussion applies to Geiger counters and to the other detection devices.

Since the theory of Geiger counter action will be discussed in considerable detail in a following paper of this series, for the present purpose, it is sufficient to say that the discharge in the counter forms a positive ion "sheath" about the central wire, which extinguishes the discharge. The counter is then unable to respond to a new ionizing event until this positive ion "sheath" has almost reached the outer cylinder. This process requires approximately \( 10^{-3} \) to \( 10^{-4} \) sec, depending on the dimensions of the counter and its gas contents. This represents a minimum insensitive period for the system. If a "gasser" amplifier circuit is used, or if the following parts of the circuit give a longer insensitive time, the actual insensitive time may be much larger.

This last fact is frequently overlooked and will be considered in a later article.
FIG. 3. This figure illustrates the regularizing action of a scale-of-m circuit with a random input. The curves show the probability that the timing spacing between two successive output counts will lie between $x$ and $x + dx$ times the average spacing.

will arrive between $x$ and $(x + dx)$ is $q_n(x)dx = \text{increase in } Q_n(x)$ during $dx$.

\[
q_n(x) = \frac{d}{dx} Q_n(x) = \sum_{m=1}^{\infty} [\ln P_{m-1}(mx) - \ln P_m(mx)]
\]

Thus $q_n(x)dx = mP_{m-1}(mx)dx$ 

\[
(mx)^{1-m}e^{-mx}dx = m^{-1}(m - 1) dx
\]

The probability distribution $q_n(x)$ of the timing spacings about the mean is thus the same as equation 16 for the inverse counting problem with order $(n - 1)$ rather than $m$. The distribution has already been shown to be normalized, equation 17, and to have a mean spacing $mx = [(m - 1) + mx]$ or $x = 1$ from equation 18. From equation 19, the most probable spacing is $mx = (m - 1)$ or $x = (1 - 1/m)$. The standard deviation of $x$, from equation 21, is $\sigma$ of $mx = \sqrt{m}$ or $\sigma$ of $x$ is

\[
\sigma = mx^{-1/2}
\]

Thus, for a scale-of-64 circuit, the standard deviation of the spacings from their mean is $\sqrt{6}$ of the value of the mean spacing. If the scaling factor $m$ is not too small, the Gaussian distribution will approximately hold and Fig. 2 may be used. Curve B of Fig. 2 gives the probability that the actual spacing will deviate from the mean spacing by $t$ times the standard deviation. Only deviations where the spacing is too small give counting losses, so half of the value from curve B should be used to calculate losses.

In Fig. 3, $mP_{m-1}(mx)$ is plotted as a function of $x$ for $m = 1, 2, 4, 8, 16$. These curves show directly the shape of the regularizing curves for scales of 1 (no scaling), 2, 4, 8, and 16. The curves are normalized so the probability for the next count to arrive during $x \leq x_1$ is just the area under the curve for $x \leq x_1$. Thus less than $1\%$ of the counts will have arrived by $x = \frac{1}{2}$ for a scale of 16, and about $10\%$ for a scale of 8. The small area of the curves up to small values of $x$ for $m > 1$ is in strong contrast with $P_n(x)$ which corresponds to no scaling action. A study of Fig. 3 shows the great value to be gained by the use of even small scaling factors when the use of large scaling factors is considered to be too expensive.

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