Chapter 4

Magnetization, Relaxation and the Bloch Equation

Chapter Contents

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Summary: The interactions of spins with their surroundings are modeled in the presence of external field effects by the phenomenological Bloch equation. The relaxation decay times $T_1$, $T_2$, $T_2'$ and $T_2^*$ are introduced. Solutions of the Bloch equation are given for constant and harmonic fields.

Introduction

Thus far, the response of an isolated proton's spin in an external magnetic field has been modeled by the classical equations of motion of a single magnetic moment. The interactions of the proton spin with its neighboring atoms lead to important modifications to this behavior. The local fields change the spin precession frequency, and the proton can exchange spin energy with the surroundings. In this chapter, we model these effects, as guided by experiment, after introducing the average magnetic dipole moment density ('magnetization').

4.1 Magnetization Vector

For images of a macroscopic body, we focus on protons, introducing their local magnetic moment per unit volume, or magnetization, as $\vec{M}(\vec{r}, t)$. Consider a volume element ('voxel')
with volume $V$ small enough that external fields are to a good approximation constant over $V$, but big enough to contain a large number of protons. The magnetization is

$$\vec{M} = \frac{1}{V} \sum_{\text{protons in } V} \vec{\mu}_i$$

(4.1)

The set of spins in $V$ is called a spin 'isochromat,' which can be defined to be an ensemble or domain of spins with the same phase. With the neglect of the proton interactions with their environment, a sum over the equations of motion for the individual spins (2.24) yields

$$\frac{1}{V} \sum_i \frac{d\vec{\mu}_i}{dt} = \frac{\gamma}{V} \sum_i \vec{\mu}_i \times \vec{B}_{ext}$$

(4.2)

or

$$\frac{d\vec{M}}{dt} = \gamma \vec{M} \times \vec{B}_{ext} \quad \text{(non-interacting protons)}$$

(4.3)

It is most advantageous to analyze the magnetization, and its differential equation, in terms of parallel and perpendicular components defined relative to the static main magnet field, $\vec{B}_{ext} = B_0 \hat{z}$. The parallel, or 'longitudinal' component of the magnetization is

$$M_\parallel = M_z$$

(4.4)

The transverse components are

$$\vec{M}_\perp = M_x \hat{x} + M_y \hat{y}$$

(4.5)

The corresponding components of the cross product in (4.3) lead to decoupled equations

$$\frac{dM_z}{dt} = 0 \quad \text{(non-interacting protons)}$$

(4.6)

and

$$\frac{d\vec{M}_\perp}{dt} = \gamma \vec{M}_\perp \times \vec{B}_{ext} \quad \text{(non-interacting protons)}$$

(4.7)

The modeling of the proton interactions with its neighborhood leads to additional terms in (4.6) and (4.7) which depend on decay parameters, and these parameters are different in the two equations. This difference is related to the fact that, in contrast to a given magnetic moment, the magnitude of the macroscopic magnetization is not fixed, since it is the vector sum of (many) proton spins. The components of $\vec{M}$ parallel and perpendicular to the external field 'relax' differently in the approach to their equilibrium values.

### 4.2 Spin-Lattice Interaction and Regrowth Solution

Equation (4.6) is certainly wrong for interacting protons, insomuch as their moments try to align with the external field through the exchange of energy with the surroundings. To understand the origin of the missing term, an energy argument is helpful.

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1The external fields are assumed to vary spatially only over scales much larger than $V^{\frac{1}{3}}$. 
4.2. Spin-Lattice Interaction and Regrowth Solution

The classical formula for the potential energy associated with a magnetic moment immersed in a magnetic field is (Appendix A)

$$U = -\vec{\mu} \cdot \vec{B}$$

(4.8)

This implies that the moment will tend to line up parallel to the field in order to reach its minimum energy state, if energy can be transferred away. Since the protons are considered to be in thermal contact with the lattice of nearby atoms, the thermal motion present in the lattice can account for any change in a given proton spin energy (4.8). In the quantum language developed in the next chapter, a spin can exchange a quantum of energy with the lattice.

The magnetization version of (4.8) is the potential energy density

$$U_M = -\vec{M} \cdot \vec{B} = -M_\parallel B_0$$

(4.9)

involving only the longitudinal component of the magnetization. Although the transverse components can be ignored in discussing the energy, it follows that, as the longitudinal magnetization returns to its equilibrium value $M_0$, the transverse magnetization must vanish. (In fact, the transverse magnetization can vanish more quickly due to `dephasing,' see the next section.) The equilibrium value relevant to room temperatures obeys Curie’s law in its dependence on the absolute temperature $T$ and the external field,

$$M_0 = C \frac{B_0}{T}$$

(4.10)

The constant $C$ is derived in Ch. 6 for protons as well as for other particles with different spins.

It is helpful to preview some of the discussion in Ch. 6 with respect to (4.10). In the applications to MRI, (4.10) is very small compared to the maximum possible magnetization (which would be the product of the spin density times the individual spin magnetic moment). Since the proton spin energy (4.8) is tiny compared with the thermal energy scale $kT$ ($k$ is the Boltzmann’s constant and $T$ is in Kelvin) at room temperature, there is only a minuscule energy advantage for a spin moment to be aligned with the magnetic field. In consequence, only a very small fraction (about five in one million for a field strength of 1.5 T) of parallel spins exceed anti-parallel spins for field strengths of interest. Fortunately, Avogadro’s number is so large, that, for example, on the order of $10^{18}$ excess proton spins are aligned along a 0.5 T field in one mole of water (see Ch. 6). Hence the magnetization $M_0$ is still big enough to be measured.

Suppose the equilibrium magnetization of a body is disturbed (by, say, the temporary application of an rf pulse) from its equilibrium value. As a result of the continued presence of the static field, the magnetization returns to its equilibrium magnetization vector $M_0 \hat{z}$. In the remainder of this section, the relaxation of the longitudinal component to $M_0$ is discussed, and, in the next section, the relaxation of the transverse components to zero is described.

**Introduction of $T_1$**

A constant interaction growth rate from the proton interactions with the lattice (see Ch. 6), implies that the rate of change of the longitudinal magnetization, $dM_\parallel(t)/dt$, is proportional
to the difference $M_0 - M_z$. The proportionality constant is empirically determined, and represents the inverse of the time scale of the growth rate. Equation (4.6) is replaced by\(^2\)

$$\frac{dM_z}{dt} = \frac{1}{T_1} (M_0 - M_z) \quad (\vec{B}_{\text{ext}} \parallel \hat{z}) \quad (4.11)$$

where $T_1$ is the experimental 'spin-lattice relaxation time.' The relaxation parameter $T_1$ ranges from tens to thousands of milliseconds for protons in human tissue over the $B_0$ field strengths of interest (0.01 T and higher.) Typical values for various tissues are shown in Table 4.1.

<table>
<thead>
<tr>
<th>Tissue</th>
<th>$T_1$ (ms)</th>
<th>$T_2$ (ms)</th>
</tr>
</thead>
<tbody>
<tr>
<td>gray matter (GM)</td>
<td>950</td>
<td>100</td>
</tr>
<tr>
<td>white matter (WM)</td>
<td>600</td>
<td>80</td>
</tr>
<tr>
<td>muscle</td>
<td>900</td>
<td>50</td>
</tr>
<tr>
<td>cerebrospinal fluid (CSF)</td>
<td>4500</td>
<td>2200</td>
</tr>
<tr>
<td>fat</td>
<td>250</td>
<td>60</td>
</tr>
<tr>
<td>blood</td>
<td>1200</td>
<td>100-200(^3)</td>
</tr>
</tbody>
</table>

Table 4.1: Representative values of relaxation parameters $T_1$ (see Sec. 4.2) and $T_2$ (see Sec. 4.3), in milliseconds, for hydrogen components of different human body tissues at $B_0 = 1.5$ T and 37 °C (human body temperature). These are only approximate values; see Ch. 22.

**Problem 4.1**

Derive (4.12) by solving the first-order differential equation (4.11).

Hint: One method is to use an integrating factor. Another is simply to put the magnetization and time variables on the opposite sides of the equation and integrate.

The solution of (4.11) can be found, for example, by the procedures outlined in Prob. 4.1. After the application of an rf pulse, the longitudinal magnetization displays an exponential form showing the evolution from the initial value, $M_z(0)$, to the equilibrium value, $M_0$:

$$M_z(t) = M_z(0)e^{-t/T_1} + M_0(1 - e^{-t/T_1}) \quad (\vec{B}_{\text{ext}} \parallel \hat{z}) \quad (4.12)$$

We reiterate that this solution corresponds to the situation where $\vec{B} = B_0\hat{z}$ and $M_0$ is the equilibrium value.\(^4\) The solution for an arbitrary starting point will be of value in multiple applications.

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\(^2\)The absence of an explicit field dependence in the solution should not be misunderstood. It is only because of the external field that the longitudinal magnetization evolves to $M_0$.

\(^3\)The higher value pertains to arterial blood and the lower value to venous blood.

\(^4\)If the external field were not uniform, the solution would only refer to a given point in space. The dependence on $\vec{r}$ has been suppressed.
rf pulse experiments:

\[ M_z(t) = M_z(t_0)e^{-(t-t_0)/T_1} + M_0(1 - e^{-(t-t_0)/T_1}) \quad (\vec{B}_{ext} \parallel \hat{z}) \quad (4.13) \]

**Problem 4.2**

The key equation (4.12) can be used to investigate general questions. If unmagnetized material is placed in a region with a finite static field at \( t = 0 \) \( (M_z(0) = 0) \):

a) Find the time it takes, in units of \( T_1 \), for the longitudinal magnetization to reach 90\% of \( M_0 \).

b) Find an approximate formula for \( M_z(t) \) of this material in the limit that \( t \ll T_1 \). Use this formula to find the initial \( (t = 0) \) slope of \( M_z(t) \), and compare the answer to the general formula indicated in Fig. 4.1, when \( M_z(0) = 0 \).

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*Formula (4.12) is the key to understanding the regrowth, after an initial disturbance, of longitudinal magnetization.* Throughout the discussion in later chapters of this book, it is often necessary to determine how much longitudinal magnetization is available to be rotated back into the transverse plane by a given sequence of rf pulses. An illustration of the exponential regrowth for a given initial value is presented in Fig. 4.1a. The time scale for regrowth is seen to be determined by \( T_1 \).

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**Fig. 4.1:** (a) The regrowth of the longitudinal component of magnetization from the initial value \( M_z(0) \) to the equilibrium value \( M_0 \). (b) The decay of the magnitude of the transverse magnetization from an initial value.
4.3 Spin-Spin Interaction and Transverse Decay

An important mechanism for the decay of the transverse magnetization is as follows. Spins experience local fields which are combinations of the applied field and the fields of their neighbors. Since variations in the local fields lead to different local precessional frequencies, the individual spins tend to fan out in time, as shown in Fig. 4.2, reducing the net magnetization vector. The ‘fanning out’ is usually referred to as ‘dephasing.’ The total transverse magnetization is the vector (or complex) sum of all the individual transverse components.

![Diagram showing spin behavior](image)

Fig. 4.2: The upper sequence shows a 90° tip of a set of spins (isochromats) into the transverse plane such that they all lie along the y-axis (laboratory frame) at some instant in time, as shown in the middle figure. Precession of the individual spins in the x-y plane immediately follows (the recovery of longitudinal magnetization is ignored since the focus is on transverse magnetization dephasing effects). The lower sequence shows the same process in terms of the net transverse magnetization which decreases in magnitude during the precession because of the fanning out of the spins.

**Introduction of \( T_2 \)**

The characterization of the overall rate of reduction in transverse magnetization brings forth another experimental parameter, the ‘spin-spin’ relaxation time \( T_2 \). The differential equation (4.7) is changed by the addition of a decay rate term

\[
\frac{d\hat{M}_\perp}{dt} = \gamma \hat{M}_\perp \times \hat{B}_{\text{ext}} - \frac{1}{T_2} \hat{M}_\perp
\]  

(4.14)
4.3. Spin-Spin Interaction and Transverse Decay

The additional term leads to exponential decay of any initial value for $\tilde{M}_\perp$. This is most easily seen in the rotating reference frame, where the differential equation has a standard decay-rate form\(^5\)

\[
\left( \frac{d\tilde{M}_\perp}{dt} \right)' = -\frac{1}{T_2} \tilde{M}_\perp \quad \text{(rotating frame)}.
\]  

(4.15)

with the solution

\[
\tilde{M}_\perp(t) = \tilde{M}_\perp(0) e^{-t/T_2} \quad \text{(rotating frame)}.
\]  

(4.16)

Equation (4.16) describes the exponential decay of the magnitude $M_\perp \equiv |\tilde{M}_\perp|$ of the transverse magnetization in either the laboratory or the rotating reference frame. A sample curve for the decay is displayed in Fig. 4.1b.

Owing to the fact that the 'spin-spin' interactions include the collective dephasing effect, where no energy is lost, as well as the same spin-lattice couplings giving rise to $T_1$ effects, (4.14) corresponds to a higher relaxation rate than (4.11). Define the relaxation rates by

\[
R_1 \equiv 1/T_1 \quad \text{and} \quad R_2 \equiv 1/T_2
\]  

(4.17)

Then

\[
R_2 > R_1 \quad \text{or} \quad T_2 < T_1
\]  

(4.18)

The relaxation parameter $T_2$ is on the order of tens of milliseconds for protons in most human tissue (see Table 4.1 for a variety of $T_2$ tissue values). It is approximately constant over the $B_0$ range of interest for a given tissue. The values of $T_2$ are much shorter for solids (on the order of microseconds) and much longer for liquids (on the order of seconds).

**Introduction of $T_2^*$ and $T_2^*$**

In practice, there is an additional dephasing of the magnetization introduced by external field inhomogeneities. This reduction in an initial value of $\tilde{M}_\perp$ can sometimes be characterized by a separate decay time $T_2^*$.\(^6\) The total relaxation rate, defined as $R_2^*$, is the sum of the internal and external relaxation rates

\[
R_2^* = R_2 + R_2^*
\]  

(4.19)

In terms of an overall relaxation time $T_2^* \equiv 1/R_2^*$,

\[
\frac{1}{T_2^*} = \frac{1}{T_2} + \frac{1}{T_2^*}
\]  

(4.20)

As we shall see later, the loss of transverse magnetization due to $T_2^*$ is 'recoverable.' To the extent that $T_2^*$ effects dominate the 'fan out' shown in Fig. 4.2, an additional pulse can be designed so as to lead to a refocusing of the spins, a reversal of the dephasing caused by the external field inhomogeneities. It is possible to recover their initial phase relationship corresponding to the initial value of $\tilde{M}_\perp$. Referred to as 'creating an echo,' this process will be described in detail in Ch. 8. The intrinsic $T_2$ losses are not recoverable; they are related to local, random, time-dependent field variations.

\(^5\)Recall from Ch. 3 that there is no need to distinguish between $\tilde{M}$ and $\tilde{M}^t$.

\(^6\)There is no guarantee that local field inhomogeneities lead to an exponential signal decay, but they are assumed to do so in this discussion.
4.4 Bloch Equation and Static-Field Solutions

The differential equations (4.11) and (4.14) for magnetization in the presence of a magnetic field and with relaxation terms can be combined into one vector equation,

\[
\frac{d\mathbf{M}}{dt} = \gamma \mathbf{M} \times \mathbf{B}_{\text{ext}} + \frac{1}{T_1} (M_0 - M_z) \mathbf{\hat{z}} - \frac{1}{T_2} \mathbf{M}_\perp
\]

(4.21)

This empirical vector equation is referred to as the Bloch equation. The relaxation terms describe the return to equilibrium, but only for a field pointing along the z-axis. The quantum mechanical underpinnings of the Bloch equation are described in Chs. 5 and 6.

Let us solve the Bloch equation for the constant field case, \( \mathbf{B}_{\text{ext}} = B_0 \mathbf{\hat{z}} \). A calculation of the components of the cross product in (4.21) produces the three component equations

\[
\frac{dM_z}{dt} = \frac{M_0 - M_z}{T_1}
\]

(4.22)

\[
\frac{dM_x}{dt} = \omega_0 M_y - \frac{M_x}{T_2}
\]

(4.23)

\[
\frac{dM_y}{dt} = -\omega_0 M_x - \frac{M_y}{T_2}
\]

(4.24)

The first equation is the same as (4.11) whose solution is (4.12). For the last two equations, the relaxation terms can be easily eliminated by the change of variables, \( M_x = m_x e^{-t/T_2} \) and \( M_y = m_y e^{-t/T_1} \) (i.e., by the introduction of integrating factors). The resulting differential equations for \( m_x \) and \( m_y \) have exactly the form of the equations found, and solved, for \( \mu_x \) and \( \mu_y \) in Ch. 2.\(^7\) In terms of the original variables, the complete set of solutions is therefore

\[
M_x(t) = e^{-t/T_2} (M_x(0) \cos \omega_0 t + M_y(0) \sin \omega_0 t)
\]

(4.25)

\[
M_y(t) = e^{-t/T_2} (M_y(0) \cos \omega_0 t - M_x(0) \sin \omega_0 t)
\]

(4.26)

\[
M_z(t) = M_z(0) e^{-t/T_1} + M_0 (1 - e^{-t/T_1})
\]

(4.27)

The equilibrium or steady-state solution can be found from the asymptotic limit \( t \to \infty \) of (4.25)-(4.27). In that limit, all the exponentials vanish implying the steady-state solution

\[
M_x(\infty) = M_y(\infty) = 0, \quad M_z(\infty) = M_0
\]

(4.28)

Problem 4.3

A direct derivation of the steady-state solution, when it exists, of a system of differential equations can often be found by the following procedure. Assuming that the system evolves to constant values for large times, all time derivatives can be set to zero. The problem reduces to a system that can often be solved analytically. Use this procedure to find the steady-state solution directly from (4.22), (4.23) and (4.24), verifying (4.28).

\(^7\)See, in particular, Prob. 2.6.
The general time-dependent solution for the transverse components, (4.25)-(4.26), is seen to have sinusoidal terms modified by a decay factor. The sinusoidal terms correspond to the precessional motion discussed in Ch. 2, and the damping factor comes from the transverse relaxation effect. The magnitude $|\vec{M}|$ is not fixed: The longitudinal component relaxes from its initial value to the equilibrium value $M_0$; the transverse component rotates clockwise and it decreases in magnitude. Recall that the transverse decay time $T_2$ is in general different from (smaller than) the longitudinal decay time $T_1$. An example of the resulting left-handed 'corkscrew' trajectory for an initial magnetization lying in the transverse plane is illustrated in Fig. 4.3.

![Diagram](image)

Fig. 4.3: The trajectory of the tip of the magnetization vector showing the combined regrowth of the longitudinal magnetization and decay of the transverse components. The initial value was along the y axis and the reference frame is the laboratory.

**Phase Description**

The general solutions could also be simplified in their description by employing the complex representation of Sec. 2.3.5

$$M_+(t) \equiv M_x(t) + iM_y(t)$$  \hspace{1cm} (4.29)

With the details left to the next problem, the solution for a static field in this representation is

$$M_+(t) = e^{-i\omega_0 t - t/T_2} M_+(0)$$  \hspace{1cm} (4.30)

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8The solutions also may be described by the rotation matrices introduced in Ch. 2.
**Problem 4.4**

a) Find the differential equation for \( M_+(t) \) analogous to (2.40) and show that its solution is (4.30).

b) Show that (4.30) is equivalent to (4.25)-(4.26).

c) Repeat (a) for
\[
M_- \equiv M_x - iM_y = -i(M_y + iM_z)
\]
(4.31)

As a follow-up on the remarks of Sec. 2.3.5 about the phase of a single moment, the phase of the complex representation of the magnetization in solutions like (4.30) plays a key role in characterizing imaging signals. The generalization of (2.41) is
\[
M_+(t) = |M_+(t)|e^{i\phi(t)} = M_+(t)e^{i\phi(t)}
\]
(4.32)
noting that the magnitude of the complex magnetization (4.29) is the same as the magnitude \( M_\perp \) of the transverse vector (4.5). For the static-field solution,
\[
M_\perp(t) = e^{-i/T_2}M_\perp(0)
\]
(4.33)
and
\[
\phi(t) = -\omega_0 t + \phi(0)
\]
(4.34)
In addition, the phase is often given with reference to rotating frames where, for the static-field case in the Larmor rotating frame, it becomes a constant, \( \phi = \phi(0) \).

### 4.5 The Combination of Static and RF Fields

An rf field needs to be added to the static field in order to tip \( \vec{M} \) from its equilibrium direction. The precession of the resulting transverse component of magnetization produces its own (rotating) field, which can be detected with a nearby coil (see Ch. 7). The analysis of the resulting motion is done expeditiously, as we have shown in Ch. 3, in rotating reference frames.

#### 4.5.1 Bloch Equation for \( \vec{B}_{\text{ext}} = B_0\hat{z} + B_1\hat{x}' \)

Following Sec. 3.3.3, we add a left-circularly polarized rf field \( \vec{B}_1 \) which is at rest in the rotating frame and parallel to \( \hat{x}' \) (see Ch. 3). The total external field is
\[
\vec{B}_{\text{ext}} = B_0\hat{z} + B_1\hat{x}'
\]
(4.35)
The effective field in that frame is
\[
\vec{B}_{\text{eff}} = (B_0 - \frac{\omega}{\gamma})\hat{z} + B_1\hat{x}'
\]
(4.36)
4.5. The Combination of Static and RF Fields

To find the Bloch component equations for (4.36), the rf field $\vec{B}_1$ is assumed to have a magnitude much smaller than $B_0$. It is supposed that there still exists a z-component equilibrium value $M_0$, and decay constants $T_1$ and $T_2$. The reader is asked to show that the component Bloch equations in the primed coordinates take the form

$$\left(\frac{dM_z}{dt}\right)' = -\omega_1 M_y' + \frac{M_0 - M_z}{T_1} \tag{4.37}$$

$$\left(\frac{dM_{z'}}{dt}\right)' = \Delta \omega M_y' - \frac{M_{z'}}{T_2} \tag{4.38}$$

$$\left(\frac{dM_y}{dt}\right)' = -\Delta \omega M_{z'} + \omega_1 M_z - \frac{M_y'}{T_2} \tag{4.39}$$

with

$$\Delta \omega \equiv \omega_0 - \omega \tag{4.40}$$

We are reminded that $\omega_0$ is the Larmor frequency, $\omega_1$ is the spin frequency due to the rf field, and $\omega$ is the rf laboratory frequency of oscillation. The $\Delta \omega$ terms in the above equations are 'off-resonance' contributions. They may represent deviations from ideal conditions due to static field impurities or variations in the applied rf frequency.

**Problem 4.5**

Taking note of (3.26), demonstrate that in the primed basis (4.21) reduces to (4.37)-(4.39).

The above differential equations contain frequency terms, which combine to produce an instantaneous rotation about an effective field, and decay constant terms. On-resonance ($\omega = \omega_0$) precession (with frequency $\omega_1 = \gamma B_1$) around $\vec{B}_1$ of the components transverse to $\vec{x}$ is evidently superimposed on the relaxation decay in (4.37) and (4.39). In the original unprimed frame, this is a nutation superimposed on the decay.

### 4.5.2 Short-Lived RF Pulses

The rf pulses in most MR measurements are designed to have a very small time duration, $\tau_{rf}$. Furthermore, the typical values for $\omega_1$ are much greater than the decay rates $1/T_1$ and $1/T_2$. The solution of the differential equations (4.37)-(4.39) can therefore be carried out in two steps, each of which is already familiar to us.

In the first step, the relaxation terms can be ignored relative to the frequency terms. Upon the replacement $\vec{M} \rightarrow \vec{\mu}$, the resulting equations are identical to (3.26), and the simple solutions of (3.26) can be employed in the present case.

In the second step, the rf pulse is considered to be turned off ($\omega_1 = 0$). Now the equations are just what was solved in the previous section, aside from a transformation to the rf rotating

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9For many imaging applications, it is also turned off a large fraction of the time.
reference frame. Upon the replacement \( \omega_0 \to \Delta \omega \) in (4.25)-(4.27), the motion in the rotating frame is described by

\[
\begin{align*}
M_x'(t) &= e^{-t/T_2} (M_x(0) \cos \Delta \omega t + M_y(0) \sin \Delta \omega t) \\
M_y'(t) &= e^{-t/T_2} (M_y(0) \cos \Delta \omega t - M_x(0) \sin \Delta \omega t) \quad (4.41) \\
M_z(t) &= M_z(0) e^{-t/T_1} + M_0 (1 - e^{-t/T_1}) \quad (4.43)
\end{align*}
\]

### 4.5.3 Long-Lived RF Pulses

The rf field is kept on for a relatively long time in some applications. The sample is then said to be 'saturated' and the long-term behavior of the magnetization can be described by steady-state solutions.

#### Steady State

To find the steady-state solutions, all time derivatives are set equal to zero in (4.37)-(4.39). We shall first derive them in the limit, \( \omega_1 = 0 \). In this limit, the steady-state solution is easily seen to be

\[
M_z^{ss} = M_0, \quad M_x^{ss} = M_y^{ss} = 0 \quad (4.44)
\]

The above zeroth-order (in \( \omega_1 \)) solution implies that as \( \omega_1 \to 0 \) the transverse magnetization components must vanish. Hence, for small but nonzero \( \omega_1 \), \( M_x' \) and \( M_y' \) must be \( \mathcal{O}(\omega_1) \) (i.e., at most first-order in \( \omega_1 \)). Therefore, from (4.37) we find \( M_0 - M_z = \mathcal{O}(\omega_1^2) \). Then, correct in first-order \( \omega_1 \), the steady-state solution must satisfy

\[
\begin{align*}
M_z^{ss} &= M_0 \quad (4.45) \\
M_y^{ss} \Delta \omega - \frac{1}{T_2} M_x^{ss} &= 0 \quad (4.46) \\
M_x^{ss} \Delta \omega + \frac{1}{T_2} M_y^{ss} &= M_0 \omega_1 \quad (4.47)
\end{align*}
\]

The solutions of (4.46) and (4.47) are

\[
\begin{align*}
M_x^{ss} &= M_0 \frac{\Delta \omega T_2}{1 + (\Delta \omega T_2)^2 \omega_1 T_2} \quad (4.48) \\
M_y^{ss} &= M_0 \frac{1}{1 + (\Delta \omega T_2)^2 \omega_1 T_2} \quad (4.49)
\end{align*}
\]

correct to \( \mathcal{O}(\omega_1^2) \). The general-order solutions are treated in a problem.

\[\text{---} \]

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\[\text{---} \]
Problem 4.6

Return to the Bloch equations (4.37)-(4.39) and solve them in the steady state for arbitrary $B_1$, obtaining

\[ M_{x'}^{ss} = M_0 \frac{\Delta \omega T_2}{D} \omega_1 T_2 \]  
\[ M_{y'}^{ss} = M_0 \frac{1}{D} \omega_1 T_2 \]  
\[ M_z^{ss} = M_0 \frac{1 + (\Delta \omega T_2)^2}{D} \]  

with

\[ D = 1 + (\Delta \omega T_2)^2 + \omega_1^2 T_1 T_2 \]

Notice that these reduce to (4.48), (4.49), and (4.45), respectively, for small $\omega_1$. In particular, show that $M_{x'}^{ss}$ and $M_{y'}^{ss}$ are $O(\omega_1)$ and that $M_0 - M_z^{ss} = O(\omega_1^2)$, consistent with the previous discussion. Also, show that the steady-state magnetization develops a phase shift in the $x$-$y$ plane whose magnitude is

\[ |\Delta \phi| = \cot^{-1}(\Delta \omega T_2) \]
Suggested Reading

The basic concepts of NMR appear in the following early papers:


