The Noether theorem

We already saw that if q is a cyclic variable, the associated conjugate momentum is conserved,

$$\frac{\partial \mathcal{L}}{\partial q} = 0 \quad \Rightarrow \quad p \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}} = \text{const} \ . \tag{1}$$

This is the simplest incarnation of Noether's theorem, which states that whenever we have a *continuous symmetry* of Lagrangian, there is an associated *conservation law*. By "symmetry" we mean any transformation of the generalized coordinates q, of the associated velocities $\dot{q}$, and possibly of the time variable t, that leaves the value of the Lagrangian unaffected. By "continuous symmetry" we mean a symmetry with a continuous constant parameter, typically infinitesimal, say ϵ, that we can dial, and that measures how far from the identity the transformation is bringing us. In a sense ϵ measures the "size" of the transformation.

In the case of the cyclic coordinate discussed above, the corresponding symmetry is simply

$$q(t) \to q(t) + \epsilon \ , \qquad \dot{q}(t) \to \dot{q}(t) \ , \qquad t \to t \ , \tag{2}$$

that is, an infinitesimal shift of the cyclic coordinate. Indeed, if we perform these replacements in the Lagrangian, at first order in ϵ the Lagrangian changes by

$$\delta\mathcal{L} \equiv \mathcal{L}(q+\epsilon, \dot{q}; t) - \mathcal{L}(q, \dot{q}; t) \simeq \frac{\partial \mathcal{L}}{\partial q}\epsilon \ , \tag{3}$$

which vanishes if an only if q is cyclic.

Theorem: Consider a Lagrangian system with n degrees of freedom $q_1, \ldots, q_n$. *If* for certain functions $\gamma_\alpha(t)$ and for constant infinitesimal ϵ the transformation

$$q_\alpha(t) \to q_\alpha(t) + \epsilon\gamma_\alpha(t) \ , \qquad \dot{q}_\alpha(t) \to \dot{q}_\alpha(t) + \epsilon\dot{\gamma}_\alpha(t) \ , \qquad t \to t \ , \tag{4}$$

is a symmetry, i.e. if it leaves the Lagrangian unaffected, *then* the quantity

$$\sum_{\alpha=1}^{n} \frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\gamma_\alpha \tag{5}$$

is a constant of motion, i.e. it is conserved. Notice that in (30) we are not transforming the time variable. We will treat the case of time transformations, in particular of time translations, separately below.

Proof: By definition of symmetry, the change in the Lagrangian upon the replacements (30) must vanish

$$\delta\mathcal{L} \equiv \mathcal{L}(q_\alpha + \epsilon\gamma_\alpha, \dot{q}_\alpha + \epsilon\dot{\gamma}_\alpha; t) - \mathcal{L}(q_\alpha, \dot{q}_\alpha; t) = 0 \ . \tag{6}$$

At first order in ϵ, this equation becomes

$$\sum_\alpha \left[\frac{\partial \mathcal{L}}{\partial q_\alpha}\epsilon\gamma_\alpha + \frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\epsilon\dot{\gamma}_\alpha\right] = 0 \ . \tag{7}$$

We can rewrite the first term by using the equations of motion:

$$\frac{\partial \mathcal{L}}{\partial q_\alpha} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha} , \qquad \forall \alpha \ . \tag{8}$$

We are left with

$$\sum_\alpha \left[\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\epsilon\gamma_\alpha + \frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\epsilon\dot{\gamma}_\alpha\right] = 0 \ . \tag{9}$$

The l.h.s. we can rewrite as a total time derivative

$$\epsilon\frac{d}{dt}\Big(\sum_\alpha \frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\gamma_\alpha\Big) = 0 \tag{10}$$

This implies that the quantity (5) is conserved:

$$\sum_\alpha \frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\gamma_\alpha = \text{const} \ . \tag{11}$$

□

Examples

Euclidean translations. Consider N point particles, interacting via a potential. The Lagrangian is

$$\mathcal{L} = \sum_{a=1}^{N} \tfrac{1}{2} m_a \dot{\vec{r}}_a^{\,2} - V(\vec{r}_1, \ldots, \vec{r}_N) \ . \tag{12}$$

If the potential only depends on the relative positions $\vec{r}_a - \vec{r}_b$, and not on the absolute ones (i.e., if there are no *external* forces),

$$V = V(\vec{r}_1 - \vec{r}_2, \ldots, \vec{r}_1 - \vec{r}_N, \vec{r}_2 - \vec{r}_3, \ldots) \ , \tag{13}$$

then *overall translations* of the system are a symmetry. An infinitesimal translation of length ϵ in an arbitrary direction $\hat{n}$ takes the form

$$\vec{r}_a \to \vec{r}_a + \epsilon\hat{n} \ , \qquad \dot{\vec{r}}_a \to \dot{\vec{r}}_a \ , \qquad t \to t \qquad \forall a \ . \tag{14}$$

The corresponding conserved quantity is thus

$$\sum_\alpha \frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\gamma_\alpha = \sum_{a=1}^{N}\sum_{i=1}^{3} \frac{\partial \mathcal{L}}{\partial \dot{r}_a^i}\hat{n}^i = \text{const} \tag{15}$$

(the role of the γ_α's is played by the cartesian components of $\hat{n}$.) The direction $\hat{n}$ is the same for all particles and we can thus pull it out of the sum over a:

$$\sum_{i=1}^{3} \hat{n}^i \sum_{a=1}^{N} \frac{\partial \mathcal{L}}{\partial \dot{r}_a^i} = \text{const} \tag{16}$$

From (12) we have

$$\frac{\partial \mathcal{L}}{\partial \dot{r}_a^i} = m_a \dot{r}_a^i \tag{17}$$

so that

$$P^i \equiv \sum_{a=1}^{N} \frac{\partial \mathcal{L}}{\partial \dot{r}_a^i} \tag{18}$$

is nothing but the i-th component of the total momentum of the system. Our conservation law thus takes the form

$$\hat{n} \cdot \vec{P} = \text{const} \ . \tag{19}$$

Since $\hat{n}$ is an arbitrary direction, the whole vector $\vec{P}$ should be constant

$$\vec{P} = \vec{\text{const}} \ . \tag{20}$$

We therefore see that the conservation of the total momentum in the absence of external forces is a direct consequence of the invariance of the Lagrangian under spacial translations.

Euclidean rotations. If we make the further assumption that the potential only depends of the mutual *distances* $|\vec{r}_a - \vec{r}_b|$ between the particles, and not on the orientation of the relative position vectors $\vec{r}_a - \vec{r}_b$,

$$V = V(|\vec{r}_1 - \vec{r}_2|, \ldots, |\vec{r}_1 - \vec{r}_N|, |\vec{r}_2 - \vec{r}_3|, \ldots) \ , \tag{21}$$

then the Lagrangian is also invariant under *overall rotations* of the system, because the potential is, and the kinetic energy is also since it only involves the scalar quantities $\dot{\vec{r}}_a{}^2$. An infinitesimal rotation of angle ϵ about an arbitrary axis $\hat{n}$ takes the form

$$\vec{r}_a \to \vec{r}_a + \epsilon\, \hat{n} \times \vec{r}_a \ , \qquad \dot{\vec{r}}_a \to \dot{\vec{r}}_a + \epsilon\, \hat{n} \times \dot{\vec{r}}_a \ , \qquad t \to t \qquad \forall a \ . \tag{22}$$

The corresponding conserved quantity is

$$\sum_\alpha \frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha} \gamma_\alpha = \sum_{a=1}^{N} \sum_{i=1}^{3} \frac{\partial \mathcal{L}}{\partial \dot{r}_a^i} (\hat{n} \times \vec{r}_a)^i = \text{const} \tag{23}$$

(the role of the γ_α's is played by the cartesian components of $(\hat{n} \times \vec{r}_a)$.) Using (17) we get

$$\sum_{a=1}^{N} m_a \dot{\vec{r}}_a \cdot \big(\hat{n} \times \vec{r}_a\big) = \text{const} \ . \tag{24}$$

For any three vectors $\vec{A}, \vec{B}, \vec{C}$, the following identity holds:

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) \tag{25}$$

We can thus rewrite our conservation law as

$$\sum_{a=1}^{N} \hat{n} \cdot \left(\vec{r}_a \times m_a \dot{\vec{r}}_a\right) = \text{const} . \tag{26}$$

The direction $\hat{n}$ is the same for all particles. We can then pull it out of the sum, and we recognize in the remainder the *total angular momentum* of the system

$$\vec{L} \equiv \sum_{a=1}^{N} \vec{r}_a \times m_a \dot{\vec{r}}_a . \tag{27}$$

Our conservation law becomes

$$\hat{n} \cdot \vec{L} = \text{const} , \tag{28}$$

or, since the direction $\hat{n}$ is arbitrary,

$$\vec{L} = \vec{\text{const}} . \tag{29}$$

The conservation of the total angular momentum is a direct consequence of the invariance of the Lagrangian under overall rotations of the system.

Time translations. This case has to be treated separately because our simplified formulation of the general theorem does not cover it. By "time-translation" we mean the transformation

$$q_\alpha(t) \to q_\alpha(t) , \qquad \dot{q}_\alpha(t) \to \dot{q}_\alpha(t) , \qquad t \to t + \epsilon . \tag{30}$$

That is, we do nothing to the coordinates and to the velocities, but we shift time by an infinitesimal constant. This is a symmetry if and only if the Lagrangian does not depend *explicitly* on time. Indeed, the variation of the Lagrnagian under the above transformation would be

$$\delta\mathcal{L} \equiv \mathcal{L}(q, \dot{q}; t + \epsilon) - \mathcal{L}(q, \dot{q}; t) \simeq \frac{\partial \mathcal{L}}{\partial t} \epsilon , \tag{31}$$

which vanishes if and only if the partial time-derivative of the Lagrangian vanishes. Recall that the *total* time-derivative of the Lagrangian does not vanish in general, because on any given solution the value of Lagrangian depends on time also through the time dependence of the q's and of the $\dot{q}$'s. One has

$$\frac{d}{dt}\mathcal{L} = \frac{\partial \mathcal{L}}{\partial t} + \sum_\alpha \Big[\frac{\partial \mathcal{L}}{\partial q_\alpha}\dot{q}_\alpha + \frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\ddot{q}_\alpha\Big] \tag{32}$$

We can rewrite the first term inside the brackets via the eom (8). We get

$$\frac{d}{dt}\mathcal{L} = \frac{\partial \mathcal{L}}{\partial t} + \sum_\alpha \Big[\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\dot{q}_\alpha + \frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\ddot{q}_\alpha\Big] \tag{33}$$

We then notice that the two terms inside the brackets combine to give a total time derivative:

$$\frac{d}{dt}\mathcal{L} = \frac{\partial \mathcal{L}}{\partial t} + \frac{d}{dt}\sum_{\alpha} \frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\dot{q}_\alpha \tag{34}$$

This equation is more conveniently rewritten as

$$\frac{d}{dt}H = -\frac{\partial \mathcal{L}}{\partial t} \ , \tag{35}$$

where we defined H, the *Hamiltonian* of the system, as

$$H \equiv \sum_{\alpha} \frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\dot{q}_\alpha - \mathcal{L} \ . \tag{36}$$

In summary, equation (35) is always valid, but *if* the Lagrangian is invariant under time-translations, that is if it does not depend explicitly on time, then the Hamiltonian of the system is conserved

$$H = \text{const} \ . \tag{37}$$

In most physically relevant cases the value of the Hamiltonian is the *total energy*. We thus discovered that the conservation of energy is a direct consequence of the invariance of the Lagrangian under time translations. Under stable conditions, if you perform a lab experiment today or tomorrow you expect to get the same results. This fact alone implies that energy is conserved.