Homework 2 – Solutions

Massless spin-3 particles

Consider a massless particle with helicity $\pm 3$.

1. Construct the field operator and discuss its behavior under Lorentz transformations.

2. What is the gauge invariance we need in order to construct a Lorentz invariant theory?

3. What is the “field-strength” operator in this case—the analogue of $F^{\mu \nu}$, which is gauge-invariant and therefore Lorentz-covariant?

4. By $S$-matrix arguments, we have seen that in the low-energy limit, massless spin-1 particles (“photons”) can only couple to conserved charges, and massless spin-2 particles (“gravitons”) can only couple to the four-momentum. If we go through the same argument for our massless spin-3 particles, what do we get?

Solution

1. As usual, the polarization tensors with helicity $h = \pm 3$ are

$$e_{\pm 3}^{\mu \rho} (\vec{p}) = e_{\pm 1}^{\mu} (\vec{p}) e_{\pm 1}^{\rho} (\vec{p}) e_{\pm 1}^{\nu} (\vec{p}) ,$$

where $e_{\pm 1}^{\mu} (\vec{p})$ is the polarization tensor for a photon of momentum $\vec{p}$ and helicity $\pm 1$. The field operator for our spin-3 particle is

$$C^{\mu \nu \rho} (x) \equiv \frac{1}{(2\pi)^{3/2}} \sum_{h=\pm 3} \frac{d^3 p}{\sqrt{2p^0}} \left[ e_{h}^{\mu \nu \rho} (\vec{p}) a_h (\vec{p}) e^{i p \cdot x} + e_{h}^{\mu \nu \rho \ast} (\vec{p}) a_h^\dagger (\vec{p}) e^{-i p \cdot x} \right] ,$$

Under Lorentz transformations everything transforms nicely but the polarization tensors:

$$\Lambda^\mu_{\mu'} \Lambda^\nu_{\nu'} \Lambda^\rho_{\rho'} e_{\pm 3}^{\mu' \nu' \rho'} (\vec{p}) = \cdots = e^{\pm 3i \theta} (e_{\pm 3}^{\mu \nu \rho} (\Lambda \vec{p}) + (\Lambda \vec{p})\xi_{\mu \nu \rho} + \text{permutations of } \mu, \nu, \rho) ,$$

where the phase $\theta$ and the symmetric tensor $\nu_{\mu \nu}$ depend on the transformation $\Lambda$ as well as on $\vec{p}$. As a result, the field operator transforms as

$$U (\Lambda) C^{\mu \nu \rho} (x) U^{-1} (\Lambda) = (\Lambda^{-1})^\mu_{\mu'} (\Lambda^{-1})^\nu_{\nu'} (\Lambda^{-1})^\rho_{\rho'} C^{\mu' \nu' \rho'} (\Lambda x)$$

$$+ \partial^\mu \xi_{\Lambda}^{\nu \rho} (x) + \partial^\nu \xi_{\Lambda}^{\mu \rho} (x) + \partial^\rho \xi_{\Lambda}^{\mu \nu} (x) ,$$

where $\xi_{\Lambda}^{\mu \nu} (x)$ is a symmetric linear combination of creation and annihilation operators that depends on the applied Lorentz transformation $\Lambda$. 

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2. The above transformation law tells us that in order to have Lorentz invariant dynamics, our Lagrangian should have a Lorentz invariant looking structure and also be invariant under the gauge transformation

$$C^{\mu\nu\rho}(x) \rightarrow C^{\mu\nu\rho}(x) + \partial^\mu \xi^{\nu\rho}(x) + \partial^\nu \xi^{\mu\rho}(x) + \partial^\rho \xi^{\mu\nu}(x),$$

for a generic symmetric tensor function $\xi^{\mu\nu}(x)$.

3. For the photon, we know that we can get a true tensor field if we use the polarization tensor $E^{\mu \nu}(\vec{p}) \equiv (i \vec{p}_\mu e^{\nu}_{\pm 1}(\vec{p}) - i \vec{p}_\nu e^{\mu}_{\pm 1}(\vec{p}))$, which still has helicity $\pm 1$, in place of $e^{\mu}_{\pm 1}(\vec{p})$. So here we just have to construct a field operator out of the product of three such polarization tensors,

$$E_{\pm 3}^{\mu \nu \rho \sigma \tau \omega}(\vec{p}) \equiv E_{\pm 3}^{\mu \nu}(\vec{p}) E_{\pm 1}^{\rho \sigma}(\vec{p}) E_{\pm 1}^{\tau \omega}(\vec{p})$$

$$D^{\mu \nu \rho \sigma \tau \omega} \equiv \frac{1}{(2\pi)^{3/2}} \sum_{h=\pm 3} \int \frac{d^3p}{\sqrt{2p^0}} [E_h^{\mu \nu \rho \sigma \tau \omega}(\vec{p}) a_h(\vec{p}) e^{ip \cdot x} + \text{h.c.}]$$

To get the explicit expression in terms of derivatives of the original $C^{\mu \nu \rho}(x)$, we can use the mnemonic rule we discussed for the Riemann tensor:

$$D^{\mu \nu \rho \sigma \tau \omega} \sim F^{\mu \nu} F^{\rho \sigma} F^{\tau \omega}$$

$$= (\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial^\rho A^\sigma - \partial^\sigma A^\rho)(\partial^\tau A^\omega - \partial^\omega A^\tau)$$

$$\rightarrow (\partial \partial \partial C) ([\mu][\rho][\tau][\omega]),$$

where by the last expression we mean a tensor built out of triple derivatives acting on $C$, antisymmetric under exchanging the indices within each of the bracketed pairs, and totally symmetric under exchanging the pairs. The overall normalization of course is a matter of convention. Given the symmetries of $D$, one can check that it is invariant under the gauge transformation (6), and it is therefore a true Lorentz tensor.

4. Let’s call $c$ the coupling constant of a given particle to a very soft massless spin-3 particle. That is, following the same notation of the lecture notes, let’s assume that in the very soft limit the emission amplitude is

$$\mathcal{M} = 2i (2\pi)^{4} c p^\mu p^\nu p^\rho \cdot e^*_{\mu \nu \rho}(q) \cdot \delta_{\sigma \sigma'} , \quad \text{for} \quad q^\mu \rightarrow 0.$$  

(This is our elementary vertex amplitude.) Then, the amplitude for emitting a very soft spin-3 particle in a generic $\alpha \rightarrow \beta$ process is

$$\mathcal{M} \rightarrow \mathcal{M}_{\alpha\beta} \sum_n \eta_n c_n p^\mu_n p^\nu_n p^\rho_n \cdot e^*_{\mu \nu \rho}(q)$$

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Lorentz invariance tells us that this expression should vanish if we replace the emitted particle’s polarization tensor with $q_\mu$. We get

$$\mathcal{M}_{\alpha\beta} \sum_n \eta_n c_n p^\mu_n p_\nu_n = 0$$

That is, processes with non-vanishing $\mathcal{M}_{\alpha\beta}$ must conserve the tensor

$$O^{\mu\nu} = \sum c_n p^\mu_n p_\nu_n.$$  \hspace{1cm} (16)

But, as we discussed, the only kinematical constraint obeyed by all non-trivial scattering processes is the total four-momentum conservation law. For instance, by applying the conservation of $O^{\mu\nu}$ to a $2 \rightarrow 2$ elastic scattering one gets that either the two particles do not interact at all, or the coupling constants $c_n$ must vanish. We are thus led to the conclusion that

$$c_n = 0$$  \hspace{1cm} (17)

for all particles. That is, our massless spin-3 particle cannot have any interactions that survive in the very soft limit, $q \rightarrow 0$. The same conclusion applies to higher spins. Notice that this does not mean that high spin massless particles do not exist. It means that, if they do exist, they cannot have interactions that survive at low energies. In particular, they cannot mediate long range $1/r^2$ forces like Coulomb’s or Newton’s.