1 Stress-energy tensor

For a scalar field with action
\[ S = \int d^4x \sqrt{-g} \left[ -\frac{1}{2} g^{\mu\nu} (\partial_\mu \phi \partial_\nu \phi) - V(\phi) \right] \] (1)

compute the stress-energy tensor in flat space in two ways:

1. from the GR definition
\[ T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} ; \] (2)

2. from Noether’s theorem, as the current associated with spacetime translations
\[ \phi(x^\mu) \rightarrow \phi'(x^\mu) \equiv \phi(x^\mu + \epsilon^\mu) \] (3)

and check that the two methods yield the same result.

You will need the variation of \( \sqrt{-g} \) under \( g_{\mu\nu} \rightarrow g_{\mu\nu} + \delta g_{\mu\nu} \), which you can get from
\[ \det(A + \delta A) = \exp \left[ \log \det(A + \delta A) \right] = \exp \left[ \text{Tr}(\log(A + \delta A)) \right] \approx (\det A) \left[ 1 + \text{Tr}(A^{-1} \cdot \delta A) \right] \] (4)

which holds for any invertible, real, symmetric matrix \( A_{ab} \).

**Solution**

1. In order to compute (2) from (1) we need the variation of \( g^{\mu\nu} \) and \( \sqrt{-g} \) under \( g_{\mu\nu} \rightarrow g_{\mu\nu} + \delta g_{\mu\nu} \)—these are the only two objects that depend on the metric. For the former we use that for a generic (invertible, diagonalizable, etc.) matrix \( A \) we have
\[ (A + \delta A)^{-1} = (1 - A^{-1} \cdot \delta A + \ldots) \cdot A^{-1} \] (5)

(this is just the small \( x \) expansion of \( \frac{1}{1+x} \), applied to matrices). At first order in \( \delta g_{\mu\nu} \) we thus have
\[ \delta g^{\mu\nu} = -g^{\mu\alpha} g^{\nu\beta} \delta g_{\alpha\beta} . \] (6)

For the determinant we use eq. (4), which gives us
\[ \delta g = g g^{\mu\nu} \delta g_{\mu\nu} \] (7)

and therefore
\[ \delta \sqrt{-g} = \frac{1}{2} \sqrt{-g} g^{\mu\nu} \delta g_{\mu\nu} . \] (8)
Overall we get
\[
\delta S = \int d^4x \sqrt{-g} \delta g_{\mu\nu} \left[ \frac{1}{2} \partial^\mu \phi \partial^\nu \phi + \frac{1}{2} g^{\mu\nu} \mathcal{L} \right]
\]  
(9)
where the Lagrangian $\mathcal{L}$ is the term in brackets in eq. (1). From the definition (2) we finally have
\[
T^{\mu\nu}(x) = \partial^\mu \phi \partial^\nu \phi + g^{\mu\nu} \mathcal{L}
\]  
(10)

2. To get the current associated with spacetime translations, we perform an infinitesimal transformation (3) with a spacetime dependent $\epsilon^\mu = \epsilon^\mu(x)$:
\[
\phi(x) \rightarrow \phi'(x) = \phi(x + \epsilon(x)) \simeq \phi(x) + \epsilon^\mu(x) \partial^\mu \phi(x) .
\]  
(11)
The (flat space) action is invariant under this only for constant $\epsilon^\mu$. Instead now get
\[
\delta S = \int d^4x \delta \mathcal{L}
\]
(12)
\[
\simeq \int d^4x \left[ \frac{\partial \mathcal{L}}{\partial \phi} \epsilon^\mu \partial^\mu \phi(x) + \frac{\partial \mathcal{L}}{\partial (\partial^\nu \phi)} \partial_\nu (\epsilon^\mu(x) \partial^\mu \phi(x)) \right]
\]
(13)
\[
= \int d^4x \left[ \partial_\mu \mathcal{L} \epsilon^\mu(x) + \frac{\partial \mathcal{L}}{\partial (\partial_\nu \phi)} \partial_\mu \phi \partial_\nu \epsilon^\mu(x) \right]
\]  
(14)
To use Noether’s theorem, we have to rewrite this explicitly as (minus) the integral of a ‘current’ $T^\nu_{\ \mu}$ multiplying $\partial_\nu \epsilon^\mu$ (see e.g. Weinberg QFT1, sect. 7.3). Integrating by parts the first term we get
\[
\delta S = -\int d^4x T^\nu_{\ \mu} \partial_\nu \epsilon^\mu ,
\]  
(15)
where
\[
T^\nu_{\ \mu} = -\frac{\partial \mathcal{L}}{\partial (\partial_\nu \phi)} \partial_\mu \phi + \delta^\nu_{\ \mu} \mathcal{L} = \partial^\nu \phi \partial_\mu \phi + \delta^\nu_{\ \mu} \mathcal{L},
\]  
(16)
which is clearly the flat space version of (10).

2 Constraint equations

[Wald, chapter 10, problem 5]
Show that among Einstein’s equations in vacuum, the four combinations
\[
G^\mu_{\ \nu} n^\nu = 0
\]  
(17)
are the constraints, that is, they do not involve second time-derivatives acting on the metric. $n^\mu$ is the unit normal to constant-time surfaces,
\[
n^\mu \propto \partial_\mu t(x) = (1, \vec{0}) .
\]  
(18)
Solution

First, since we are interested in second derivatives of the metric tensor, it is convenient to write the Einstein tensor explicitly in terms of derivatives of \( g_{\mu\nu} \) as (see e.g. Wald eq. (10.2.26))

\[
G_{\mu\nu} = -\frac{1}{2} g^{\alpha\beta} \left[ -\partial_\beta \partial_\nu g_{\mu\alpha} - \partial_\beta \partial_\mu g_{\nu\alpha} + \partial_\alpha \partial_\beta g_{\mu\nu} + \partial_\mu \partial_\nu g_{\alpha\beta} \right] 
+ \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} g^{\rho\sigma} \left[ -\partial_\beta \partial_\rho g_{\sigma\alpha} + \partial_\alpha \partial_\beta g_{\rho\sigma} \right] + \ldots ,
\]

(19)

\[
+ \frac{1}{2} g_{\mu\nu} \propto g^{\mu\nu} \propto g^{\nu\sigma} \left[ -\partial_\sigma \partial_\rho g_{\alpha\nu} + \partial_\nu \partial_\sigma g_{\rho\alpha} \right] + \ldots ,
\]

(20)

where the dots stand for terms involving just first derivatives of the metric—which do not concern us. Then we have to contract \( G_{\mu\nu} \) with \( n^\nu \). Apart from a normalization factor, we have

\[
n^\nu = g^{\nu\eta} n_\eta \propto g^{\nu\mu} ,
\]

(21)

because of (18). We thus have

\[
G_{\mu\nu} n^\nu \propto G_\mu \equiv -\frac{1}{2} g^{\alpha\beta} g^{\nu\sigma} \left[ -\partial_\beta \partial_\nu g_{\mu\alpha} - \partial_\beta \partial_\mu g_{\nu\alpha} + \partial_\alpha \partial_\beta g_{\mu\nu} + \partial_\mu \partial_\nu g_{\alpha\beta} \right] 
+ \frac{1}{2} \delta_\mu^0 g^{\alpha\beta} g^{\rho\sigma} \left[ -\partial_\beta \partial_\rho g_{\sigma\alpha} + \partial_\alpha \partial_\beta g_{\rho\sigma} \right] + \ldots ,
\]

(22)

Depending of the value of the \( \mu \) index:

- For \( \mu = i \), the second line vanishes (because of the \( \delta_\mu^0 \)) and the only terms in the first line that can have second time derivatives are the first and the third:

\[
G_i \supset -\frac{1}{2} g^{\alpha\beta} g^{\nu\sigma} \left[ -\partial_\beta \partial_\nu g_{i\alpha} + \partial_\alpha \partial_\beta g_{i\nu} \right] 
\supset + \frac{1}{2} g^{\alpha\beta} g^{\rho\sigma} \partial_\beta \partial_\nu g_{i\alpha} - \frac{1}{2} g^{\nu\sigma} \partial_0 \partial_0 g_{i\nu} = 0 ,
\]

(24)

(25)

(26)

where we only kept track of terms with second time derivatives.

- For \( \mu = 0 \) we get many more terms:

\[
G_0 \supset \frac{1}{2} \left\{ g^{\alpha\nu} g^{\nu\rho} \partial_0 \partial_0 g_{0\alpha} + g^{\alpha\nu} g^{\nu\rho} \partial_0 \partial_0 g_{\rho\alpha} - g^{\nu\sigma} g^{\sigma\nu} \partial_0 \partial_0 g_{0\nu} - g^{\alpha\beta} g^{\nu\sigma} \partial_0 \partial_0 g_{0\alpha} - g^{\alpha\beta} g^{\nu\sigma} \partial_0 \partial_0 g_{\nu\sigma} \right\} 
\supset -g^{\alpha\sigma} g^{\nu\rho} \partial_0 \partial_0 g_{\sigma\nu} + g^{\alpha\sigma} g^{\nu\rho} \partial_0 \partial_0 g_{\sigma\nu} = 0 .
\]

(27)

(28)

(29)

In conclusion, \( G_{\mu\nu} n^\nu \) contains no second time-derivatives of the metric.
3 ADM energy-momentum

The Schwarzschild metric

\[ ds^2 = - \left( 1 - \frac{2GM}{r} \right) dt^2 + \left( 1 - \frac{2GM}{r} \right)^{-1} dr^2 + r^2 d\Omega^2 \]  (30)

describes the outside of a spherical body of total mass \( M \), in asymptotically flat space. Check that the total four-momentum of this solution is indeed

\[ P^\mu = (M, \vec{0}) . \]  (31)

Recall that the formulae for \( P^0 \) and \( \vec{P} \) in terms of \( h_{\mu\nu} \) are given in ‘cartesian coordinates’—that is, coordinates such that the metric at infinity becomes \( \eta_{\mu\nu} \). In order to apply them to the metric (30), one has first to change coordinates. The only requirement is that the metric asymptotes to \( \eta_{\mu\nu} \). Therefore, one can just use the flat-space coordinate change from spherical to cartesian coordinates:

\[ x = r \sin \theta \cos \phi , \quad y = r \sin \theta \sin \phi , \quad z = r \cos \theta , \]  (32)

regardless of what this corresponds to close to the body.

Solution

To use our formulae for the total energy and momentum,

\[ P^0 = - \frac{1}{16\pi G} \int \left[ \partial_i h_{jj} - \partial_j h_{ij} \right] n_i r^2 d\Omega \]  (33)

\[ P^j = - \frac{1}{16\pi G} \int \left[ (-\partial_t h_{kk} + \partial_k h_{k0}) \delta_{ij} - \partial_i h_{j0} + \partial_j h_{ij} \right] n_i r^2 d\Omega , \]  (34)

we first have to convert the metric in coordinates that approach cartesian coordinates at \( r \to \infty \), that is, coordinates for which the metric approaches precisely \( \eta_{\mu\nu} = \text{diag}(-1,1,1,1) \). We can use the standard mapping (32). Calling

\[ \vec{x} \equiv (x, y, z) , \]  (35)

we have

\[ r^2 = \vec{x} \cdot \vec{x} \]  (36)

\[ (dr)^2 = \left( d\sqrt{\vec{x} \cdot \vec{x}} \right)^2 = \left( \frac{\vec{x} \cdot d\vec{x}}{r} \right)^2 \]  (37)

\[ r^2 d\Omega^2 = r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) = d\vec{x} \cdot d\vec{x} - (dr)^2 = \left( \frac{\vec{x} \cdot d\vec{x}}{r} \right)^2 \]  (38)
The metric (30) then reads
\[
ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left[\left(1 - \frac{2GM}{r}\right)^{-1} - 1\right] \left(\frac{x \cdot dx}{r}\right)^2 + (dx)^2,
\]
which manifestly approaches the flat metric for \(r \to \infty\):
\[
ds^2 \to -dt^2 + (dx)^2.
\]

We can now take the integrals that defines \(P^0\) and \(P^i\). The spatial momentum \(P^i\) is manifestly zero, because the metric does not depend on \(t\) and has vanishing \(h_{0j}\) components. As to \(P^0\), at lowest order in \(1/r\) (recall that we have to take the integral over a sphere at \(\infty\)) we have
\[
h_{ij} = (2GM) \frac{x_i x_j}{r^3}
\]
\[
\partial_k h_{ij} = (2GM) \frac{1}{r^3} \left[\delta_{ik} x_j + \delta_{jk} x_i - 3 \frac{x_i x_j x_k}{r^2}\right]
\]
Using that \(n^i = x^i/r\) and taking the angular integral we get
\[
P^0 = M.
\]