Problems:

195. French and Taylor 3-8
196. French and Taylor 4-1
197. French and Taylor 8-1

198. Consider a 1-dimensional problem in which the potential $V(x)$ approaches zero at $x \to \pm \infty$. Assume that in some region the potential $V(x)$ is negative. Show that there must be an energy eigenstate with negative eigenvalue $E < 0$. You might do this as follows:

   (a) If $\psi_n$, $0 \leq n \leq \infty$ are eigenstates with energy eigenvalues $E_n$ show that a general state $\psi = \sum_{n=0}^{\infty} c_n \psi_n$ normalized to unit length has an expectation value for the energy operator $H = \frac{p^2}{2m} + V(x)$ given by:

   $$\langle \psi | H | \psi \rangle = \sum_{n=0}^{\infty} |c_n|^2 E_n$$

   and that

   $$||\psi||^2 = \langle \psi | \psi \rangle = \sum_{n=0}^{\infty} |c_n|^2 = 1.$$  

   (b) Use this result to prove that the eigenstate $\psi_0$ is the normalized state with the lowest energy expectation value.

   (c) As in the figure, observe that a second, square well potential $V_{sq}(x)$ which obeys $V_{sq}(x) > V(x)$ for all $x$ can be constructed.

   (d) Combine these results with the fact that $H_{sq} = \frac{p^2}{2m} + V_{sq}(x)$ always has a negative energy eigenfunction to deduce that the same must be true for the original $H$. 

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Reading:

*May 2* Taylor Chapter 8

*May 3* Taylor Chapters 6 and 7 (Similar material was discussed earlier in lecture so this may now be useful.)
199. The two spin-1/2 particles with angular momenta $\vec{J}^D$ and $\vec{J}^E$ from problems 184 and 191 have gyromagnetic ratios $\gamma_D$ and $\gamma_E$. They are located at fixed, nearby positions and interact with each other and with an external magnetic field $\vec{B} = B\hat{z}$. The resulting Hamiltonian is:

$$H = -\gamma_D \vec{J}^D \cdot \vec{B} - \gamma_E \vec{J}^E \cdot \vec{B} + \alpha \vec{J}^D \cdot \vec{J}^E$$

where $\alpha$ is a constant.

(a) Show that $[H, J^{\text{tot}}_z] = 0$ and deduce that $H$ and $J^{\text{tot}}_z$ can be simultaneously diagonalized.

(b) From the four states \( \{\hat{d}_i \otimes \hat{e}_j\}_{i=1,2; j=1,2} \) find the single eigenvector which has $J^{\text{tot}}_z = +\hbar$ and compute its energy eigenvalue.

(c) Similarly find the single eigenvector which has $J^{\text{tot}}_z = -\hbar$ and compute its energy eigenvalue.

(d) Since there are two states with $J^{\text{tot}}_z = 0$, we cannot immediately deduce which combinations of these will be eigenstates of $H$. Evaluate the four matrix elements of $H$ between these two $J^{\text{tot}}_z = 0$ states and then find the eigenvalues of the resulting $2 \times 2$ matrix to calculate these final two eigenvalues of $H$.

Note: this problem may be easier to solve if you write:

$$\vec{J}^D \cdot \vec{J}^E = J^D_z J^E_z + \frac{1}{2} (J^D_x + iJ^D_y)(J^E_x - iJ^E_y) + \frac{1}{2} (J^D_x - iJ^D_y)(J^E_x + iJ^E_y).$$

200. (This is an interesting but challenging problem.) Using the uncertainty principle, estimate the maximum length of time a pencil can be balanced on its point. Simplify the problem by treating the pencil’s motion as restricted to a two-dimensional plane. Locate the pencil’s position by the angle $\theta$ that it makes with the vertical direction.

(a) Make a rough estimate of the moment of inertia, $I$, of the pencil, rotating about its point in the 2-dimensional plane.

(b) Given an initial value of $\theta = \theta_0$ and the initial angular momentum $L_0$, find the resulting angular position, $\theta(t)$ at a later time $t$ in the approximation that $\theta(t) \ll 1$. (Hint: this is similar to simple harmonic motion but now calculated about a point of unstable equilibrium.)

(c) Choose initial values $\theta_0$ and $L_0$ which are consistent with the uncertainty relation $\Delta\theta \Delta L \geq \hbar$ and yet allow the pencil to remain vertical for as long a time as possible.

(d) Roughly, how long can the pencil remain with $\theta < 0.1$?