Physics 4201 Problem Set 3
Due in class Tuesday Sept 24

3.1. This is closely based on Griffiths 2.12 and 2.14.
Consider the simple harmonic oscillator equation for eigenfunction \( n \)

\[
-\frac{\hbar^2}{2m} \frac{\partial^2 \psi_n}{\partial x^2} + \frac{1}{2} m \omega^2 x^2 \psi_n = E_n \psi_n
\]

which may be rewritten in terms of raising and lowering operators in the two
alternative forms

\[
a_- a_+ \psi_n = \left( E_n + \frac{\hbar \omega}{2} \right) \psi_n
\]

\[
a_+ a_- \psi_n = \left( E_n - \frac{\hbar \omega}{2} \right) \psi_n
\]

(a) We saw in class that if \( \psi \) solves the Schroedinger equation with energy \( E \) then \( a_+ \psi \) solves the Schroedinger equation with energy \( E + \hbar \omega \), but this does not mean \( a_+ \psi \) is correctly normalized. Please find the normalization, in
other words, find the constant \( C_n \) implied by the statement that if \( \psi_{n-1} \) is a
correctly normalized real solution of the Schroedinger equation of energy \( E_{n-1} = (n - \frac{1}{2}) \hbar \omega \) then \( C_n a_+ \psi_{n-1} \) is a correctly normalized real solution with energy \( E_n = (n + \frac{1}{2}) \hbar \omega \). Note that this problem may be done by purely algebraic
means, or by writing out the explicit definitions of \( a_\pm \) and using integration by
parts.

(b) Using purely algebraic means (no integrals) compute \( \langle x^2 \rangle \) and \( \langle p^2 \rangle \)
for the states \( \psi_0, \psi_1, \psi_2 \).

*(c) Generalize this result to obtain a formula for \( \langle x^2 \rangle \) in state \( \psi_n \).

*(d) Consider a classical harmonic oscillator with same \( m \) and \( \omega \) and assume
energy \( E_n \). Compute the time average of \( x^2 \) over one oscillator period and
compare it to the \( \langle x^2 \rangle \) obtained for the quantum particle with the same
energy

3.2 This is a combination of Griffiths 2.21 and 2.22
Consider a free particle on a ring of circumference \( L \) (so the wave function
satisfies the Schrödinger equation \( -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} = E \psi \) and is subject to the periodic boundary conditions \( \psi(x) = \psi(x + L) \)). Suppose at time \( t = 0 \) the particle is
confined to the region \( 0 < x < a \) (with \( a < L \) so the problem makes sense—indeed we will focus on the case \( a << L \)) so that \( \psi(x) = \psi_0 \) for \( 0 < x < a \) and
\( \psi(x) = 0 \) otherwise. ***Note that here \( \psi_0 \) is just a number****

Please
(a) Normalize \( \psi \).
(b) Express \( \psi \) in terms of the eigenstates of the Schrödinger equation.
(c) Write an expression for \( \psi(x,t) \) for all \( t > 0 \).
(d) Write an expression for the probability \( P_n(t) \) that at time \( t \) the particle
is in the range \( 0 < x < a \).
Please do not bother to evaluate the expressions, which are complicated.

From the first problem set, we saw that wave functions spread with time, so that we expect the probability that the particle is in the range $0 < x < a$ to decrease with time. But does this decrease continue forever?

(e) Prove that $P_n(t)$ is periodic: $P_n(t + T) = P_n(t)$ with $T = 2\pi\hbar/(2mL^2)$. Estimate $T$ in seconds for an electron in a ring of size 1mm.

3.3 The result of this problem will be useful when we consider magnetic fields and the Aharonov Bohm effect, and (although we will not get to this in this course) is fundamental to superconductivity and especially "Josephson junctions".

Consider a free particle of mass $m$ on a ring of circumference $L$ but subject to the boundary condition

$$\psi(x) = e^{i\theta}\psi(x + L)$$

with $\theta$ an arbitrary real number. Please

(a) Find the three lowest energy eigenstates (including degeneracies) and the associated eigenfunctions.

(b) Plot the evolution of the ground state and first excited state energies as $\theta$ is varied from 0 to $2\pi$.

3.4 This is just Griffiths 2.20 and is also (you will notice) essentially a repetition of what was discussed in class. It is important that you understand this, so I would like you to go through the steps yourselves.

Dirichlet’s theorem says that ‘any’ function on the interval $[-a,a]$ can be expanded as a Fourier series:

$$f(x) = \sum_{n=0}^{\infty} a_n \sin(n\pi x/a) + b_n \cos(n\pi x/a)$$

Show that this can be rewritten as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/a}$$

and find $c_n$ in terms of $a_n$ and $b_n$. Note that unlike in class we are not assuming any particular boundary conditions.

(b) Find $c_n$ in terms of $f(x)$.

(c) Eliminate $c_n$ and $n$ in favor of the new variables $k = n\pi/a$ and $F(k) = \sqrt{2/\pi} a c_n$. Write the equations for $f(x)$ and $c_n$ in terms of these new variables.

(d) Take the limit $a \to \infty$ to obtain an integral representation of $f(x)$ in terms of $F(k)$ and an integral representation of $F(k)$ in terms of $f(x)$. 

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