Chapter 1

OPTICAL CONDUCTIVITY AND CORRELATED ELECTRON PHYSICS

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Abstract  This article reviews the uses of optical conductivity, and in particular optical spectral weight in elucidating the physics of correlated electron physics. An introduction to the fundamental theoretical concepts is given, followed by a summary of results obtained on specific models and a discussion of available data.

Keywords: Optical Conductivity; Correlated Electron Systems; High Temperature Superconductivity; Colossal Magnetoresistance; Heavy Fermion Compounds

1. Introduction

The optical conductivity \( \sigma_{ab}(q, \omega) \) is the linear response function relating the current, \( j \), in the \( a \) direction to an applied transverse electric field, \( E \), in the \( b \) direction:

\[
j_a(q, \omega) = \sigma_{ab}(q, \omega) E_b(q, \omega)
\]  (1.1)

One may approximately distinguish two kinds of optical processes: promotion of an electron from one orbital to another on the same ion (as in the \( 1s \rightarrow 2p \) hydrogenic transition familiar from elementary atomic physics) or motion of an electron from one place in the sample to another. The latter contribution means that the conductivity contains information on the ease with which electrons may move through the solid, and

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is of interest because the essence of the 'strong correlations' problem is the interplay between the localizing effect of repulsive electron-electron interactions and the delocalizing effect of wave function hybridization.

To measure the conductivity one must apply a transverse electric field to the sample. For most frequencies of interest this may be accomplished by exposing the sample to electromagnetic radiation; indeed the conductivity is usually inferred by exposing a material sample to electromagnetic radiation and measuring the resulting reflection and transmission properties. The frequency scales of interest for 'correlated electron' physics are typically several electron volts or less. At these frequencies the magnetic field component of the incoming radiation is negligible and one may think of the applied radiation as corresponding to a transverse electric field only. Although electromagnetic fields are transverse in free space, a longitudinal component may be generated inside a solid. In most situations of present interest, the symmetry is such that this does not occur, and the possibility will be mostly be neglected here (an important exception of current interest concerns 'bilayer plasmons' in the c-axis conductivity of high-$T_c$ materials; for a discussion and other references see e.g. [1]). Finally, the radiation wavelength is in almost all cases very large relative to any relevant length scale in the solid; thus one usually sets $q \to 0$ and discusses $\sigma(\omega)$. (Note that the crucial length scale is not the free space wavelength $\lambda = c/(2\pi \omega)$ but rather the penetration depth $\delta \sim \sqrt{1/\omega \sigma}$ [2] characterizing the decay of incident radiation inside the solid. In the 'correlated electron' materials of interest the strong correlations themselves constrain the motion of electrons, leading normally to a small conductivity and thus to a sufficiently long $\delta$ that one may neglect the spatial variation of the field inside the solid).

We henceforth consider the $q \to 0$ limit, introducing the vector potential $A$ by writing $\mathbf{E}(\omega) = \frac{\mathbf{i} \omega}{c} A(\omega)$:

$$j_a(\omega) = \sigma_{ab}(\omega) \frac{A_b(\omega)}{i \omega / c} \quad (1.2)$$

This paper will review the theory of $\sigma(\omega)$ (with particular emphasis on sum rules), show how measurements of $\sigma$ have been used to elucidate some aspects of the physics of presently interesting materials including high-$T_c$ cuprates and the 'colossal' magnetoresistance manganites, outline some open issues and mention desirable improvements in experimental technique. Almost all of the specific results presented may be found in the literature somewhere, but I hope a unified presentation will be useful. The reader's attention is also drawn to Chapter 7, which treats the optical conductivity of high-$T_c$ superconductors and touches on many of issues considered here.
2. Theory

2.1 Fundamental definitions

2.1.1 Basics. We are interested in the current induced in a system to a transverse applied electric field. To compute this one needs a Hamiltonian and a coupling to the electric field. The most fundamental Hamiltonian and coupling one would write is

$$H = \int d^3r \frac{1}{2m} \left( \nabla - \frac{ie}{c} A(r) \right)^2 + H_{\text{el-ion}} + H_{\text{el-el}} + H_{\text{ion}}$$  \hspace{1cm} (1.3)

where $\psi$ is the electron field operator, $m$ is the electron mass, and we include electron-ion and electron-electron interactions. We have not written the coupling of the electric field to the ions explicitly because the very large (relative to $m$) ion mass means that at any reasonable electron density the ion contribution to $\sigma$ is negligible (exceptions can occur for materials with a very strong electronic anisotropy: the c-axis conductivity of high-$T_c$ materials is an important example). The way the vector potential enters is dictated by gauge invariance.

The electrical current density operator $\vec{j}$ is defined in general as

$$\vec{j} = \frac{e}{V} \frac{\delta H}{\delta A}$$  \hspace{1cm} (1.4)

where $V$ is the volume of the sample. From Eqs (1.3), (1.4) one finds the familiar expression

$$\vec{j} = \int d^3r \left( \frac{ie}{m} \left( \psi^*(r) \nabla \psi(r) - \nabla \psi^*(r) \psi(r) \right) - \frac{ie}{mc} \mathcal{A} \psi^*(r) \psi(r) \right)$$  \hspace{1cm} (1.5)

and then using the usual Kubo formulae for linear response along with the relation between $E$ and $A$ gives [3]

$$\sigma_{ab}(\omega) = \frac{ie^2}{m(\omega + i\varepsilon)} \delta_{ab} + \frac{i\chi_{ab}(\omega)}{\omega + i\varepsilon}$$  \hspace{1cm} (1.6)

where the current-current correlation function $\chi$ may be written in terms of the exact eigenstates $|n\rangle$, energies $E_n$ and partition function $Z$ as

$$\chi_{ab}(\omega) = \sum_{n,m} \frac{e^{-\beta E_n}}{Z} \left[ \frac{\langle n|j_a|m\rangle \langle m|j_b|n\rangle}{\omega - (E_m - E_n) + i\varepsilon} - \frac{\langle n|j_b|m\rangle \langle m|j_a|n\rangle}{\omega + (E_m - E_n) + i\varepsilon} \right]$$  \hspace{1cm} (1.7)

The first term in Eq (1.7) comes from the term proportional to $A$ in Eq (1.6) and the relation $n = \frac{1}{V} \int d^3r \langle \psi^*(r) \psi(r) \rangle$ (the $\langle \rangle$ denotes thermal and quantal expectation value) has been used.
The conductivity is a causal function with real \((\sigma')\) and imaginary \((\sigma'')\) parts. The real part corresponds to transitions in which energy is absorbed or removed. The real and imaginary parts are related by a Kramers-Kronig relation which may be written \((P\) denotes principal part)

\[
\sigma''(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dx}{\omega - x} \sigma'(x)
\]

(1.8)

2.1.2 Charge Stiffness. The position, \(\vec{R}\), of a charged particle which is freely accelerated by a spatially uniform electric field \(\vec{E}\) obeys the equation of motion

\[
m \frac{d^2 \vec{R}}{dt^2} = e \vec{E}
\]

(1.9)

so after Fourier transformation on time the current \(\vec{j} = e \partial \vec{R}/\partial t\) carried by a density \(n\) of such particles becomes

\[
\vec{j} = \frac{ne^2}{m(\omega + i\varepsilon)} \vec{E} = ne^2 \left( P \left( \frac{i}{\omega m} \right) + \frac{\pi}{m} \delta(\omega) \right) \vec{E}
\]

(1.10)

In other words, particles which may be freely accelerated by an applied field lead to a delta function contribution to the real part of the conductivity.

One may write in general

\[
\sigma_{\alpha\beta}'(\omega) = \frac{i e^2 D_{\alpha\beta}}{\omega + i\varepsilon} + \sigma_{\alpha\beta,\text{reg}}'(\omega)
\]

(1.11)

where \(\lim_{\omega \to 0} \omega \sigma_{\alpha\beta,\text{reg}}'(\omega) = 0\). The quantity \(D\) is the Drude weight or charge stiffness and is defined by this equation. Comparison to Eq (1.10) shows that \(D\) measures the ability of the electrons in a given system to freely accelerate in an applied electric field.

In a system without translation invariance (e.g., a disordered system), \(D = 0\). In a system with at least a discrete translation invariance (e.g., electrons in an ideal periodic lattice) at \(T = 0\) (so there is no thermally induced disorder), one may have \(D \neq 0\) (in which case one has a metallic ground state); or \(D = 0\) (in which case one has a non-metallic ground state). This classification of metals and insulators in perfectly translation invariant environments is due to Kohn \([4]\). The quantity \(D\) is the order parameter characterizing the metallic state (in a clean system) and one may think of a Mott transition \([5]\) as driving \(D\) to 0.
From the Kubo formula, Eq (1.6) one obtains the fundamental expression

\[ D_{ab} = \frac{n}{m} - \sum_{n,m} e^{-\beta E_n} \frac{Z}{(E_m - E_n)} \left[ \frac{\langle n | j_a | m \rangle \langle m | j_b | n \rangle}{(E_m - E_n)} - \frac{\langle n | j_b | m \rangle \langle m | j_a | n \rangle}{(E_m - E_n)} \right] \]  

(1.12)

Kohn [4] introduced a very useful alternative expression. Diagonalize the \( D \) tensor and consider a system which is periodic, with period \( L \), in a direction corresponding to a principal axis of \( D \). The system then has the topology of a torus. Introduce a flux \( \phi \) through this torus and compute the change in ground state energy \( E_0 \). Because the flux may be represented by a vector potential \( A_\phi = \phi/L \) one finds

\[ D = \frac{d^2 E_0}{d\phi^2} \bigg|_{\phi=0} \]  

(1.13)

This expression is useful for numerical computations on finite lattices, and also for certain formal developments given below. It may also be extended to non-zero temperatures by consideration of the change in free energy with respect to \( \phi \) [6]. A comment however is in order: the derivative in Eq (1.13) is defined in terms of the difference in ground state energies \( E_0(\Delta \phi) - E_0(0) \). In order for this to make sense the change \( \Delta \phi \) must be small enough that no level crossings occur. In a typical \( d \)-dimensional non-superfluid system, the spacing between low-lying levels is of order \( \Delta E \sim L^{-d} \) so one requires a \( \Delta \phi \sim 1/(DL^{d-1}) \). In \( d = 1 \) a \( \Delta \phi \) of order unity may be used to obtain energy differences numerically; in \( d > 1 \) the maximum permissible \( \Delta \phi \) vanishes rapidly as system size increases. Remarkably the \( D \) defined in this manner is still relevant to physical properties. For further discussion, see [7].

Of course, a real system never has perfect translation invariance. In a non-superfluid system the presence of weak breaking of translation invariance (for example from a low density of imperfections in the crystal), leads to a broadening of the delta function; typically into a Lorentzian, with area given approximately by \( D \) and width given by the residual scattering rate. If the residual scattering is small, the Lorentzian is reasonably well defined (although there is always some ambiguity about continuing it into \( \sigma_{\text{reg}}^\prime \) and its area may be found approximately. As the residual scattering increases, the ambiguities increase. A schematic representation of the conductivity is shown in Fig. (1.1).

2.1.3 Extended Drude Parametrization. We have seen that scattering processes (due for example to electron-impurity, electron-electron and electron-phonon scattering) may broaden the delta function of a non-superconducting metal. For example, the familiar 'Drude'
approximation [8] leads for noninteracting electrons in the presence of static, impurity scattering to

\[ \sigma_{\text{drude}}(\omega) = \frac{e^2 D}{-i\omega + \Gamma} \]  \hspace{1cm} (1.14)

with \( \Gamma \) a scattering rate and \( D \) equal to that of the pure system, up to corrections of order \((\Gamma/D)^2\). This familiar form motivates the widely used ‘extended drude’ parametrization of the conductivity, namely

\[ \sigma(\omega) = \frac{e^2 D}{-i(1 + \lambda(\omega))\omega + \Gamma(\omega)} \]  \hspace{1cm} (1.15)

with frequency dependent scattering rate \( \Gamma(\omega) \) and ‘mass enhancement’ \( \lambda(\omega) \) defined via

\[ \Gamma(\omega) = e^2 D \text{Re} [\sigma(\omega)^{-1}] \]  \hspace{1cm} (1.16)

\[ 1 + \lambda(\omega) = e^2 D \text{Im} [\sigma(\omega)^{-1}] \]  \hspace{1cm} (1.17)

Note that because \( \sigma^{-1} \) is the response function yielding the field induced by an applied current, it is causal, implying that \( \Gamma \) and \( \lambda \) are related by a Kramers-Kronig relation. Extraction of the magnitude of \( \Gamma \) and \( \lambda \) from data requires a choice for \( D \) but the functional form is independent of \( D \). Finally, it is important to be aware that although \( \Gamma \) and \( \lambda \) are always well defined, their interpretation in terms of scattering and mass renormalization depends crucially on physical assumptions, in particular that interband transitions do not contribute appreciably to the conductivity in the frequency range of interest. For a detailed discussion of the application of ‘extended drude’ ideas to high-temperature cuprate superconductors, see [9]
2.2 Spectral Weights and Sum Rules

2.2.1 Definition and f-sum rule. Spectral weight $K$ is $(2/\pi)$ times the integral of the real part of the conductivity over some frequency range:

$$K(\Omega) = \int_{0}^{\Omega} \frac{2d\omega}{\pi} \sigma'(\omega)$$

(1.18)

Taking the $\omega \to \infty$ limit of Eq (1.8) and using Eqs (1.6), (1.7) yields the familiar f-sum rule for the total spectral weight in the conductivity:

$$\frac{ne^2}{m} = K(\infty) = \int_{0}^{\infty} \frac{2d\omega}{\pi} \sigma'(\omega)$$

(1.19)

This derivation, due to Kohn [4], shows that the f-sum rule follows directly from causality and gauge invariance, and does not depend on boundary conditions or anything else. It is thus completely general, but in the solid state physics context not particularly useful, because the integral must be taken over all frequencies, so the conductivity integral on the right hand side of Eq. (1.19) includes transitions from deep core levels to very high lying continuum levels while the quantity $n$ is the total density of all electrons in the solid. Further, the generality of the sum rule means that it contains no information about the specifics of the system (for example, about what makes high-$T_c$ materials different from silicon). Important system-specific questions include 'how is the spectral weight distributed in frequency?' and 'how does the distribution of spectral weight change when temperature or other parameters vary?'.

2.2.2 Down-folded models and restricted sum rules. In condensed matter physics one usually does not wish to deal with the full complexity of a solid. Rather, one deals with an 'effective' or "down-folded" model which focuses on a (typically low energy) subset of the degrees of freedom. One obtains this model (at least notionally) by integrating out the other degrees of freedom in the solid. One widely used example (especially common in studies of transition metal oxides) is the 'tight binding + interactions model'. In many transition metal oxides, band theory suggests that the electronic states of interest lie in relatively narrow bands reasonably well separated from other bands. In this case it is reasonable to model the bands of interest via

$$H = - \sum_{i,\delta}(t^{ab}_{i}(\delta)c_{i,a}^{\dagger}c_{i+\delta,b} + H.c.) + H_{int}$$

(1.20)

Here $i$ denotes sites on a lattice, $\delta$ denotes a displacement connecting two lattice sites and $a, b$ denote orbitals of electrons on a given site. The
hopping matrix elements $t^{ab}(\delta)$ are often estimated from fits to a band theory calculation or, more correctly, from a down-folding procedure applied to a band calculation [10] and $H_{\text{int}}$ denotes interactions of various sorts. Such models are more amenable to theoretical analysis than is the full Hamiltonian, Eq. (1.3). In most cases of physical interest $t$ is negligible if $\delta$ is greater than one or two lattice constants. It is important to remember that interactions not included in conventional band calculations may in principle affect the values and indeed the form of the hopping part of the Hamiltonian. This issue has not been thoroughly studied, and in practice the procedure of obtaining hopping parameters from band theory and adding interactions is almost universally adopted.

To determine the optical conductivity of such models one requires a means of coupling an electromagnetic field to the model. The coupling usually adopted is the Peierls substitution $t(\delta) \rightarrow t(\delta)e^{i\mathbf{A} \cdot \mathbf{r}}$. This follows from the combined assumptions of gauge invariance and reasonably spatially localized wave functions, so that in the presence of a slowly spatially varying electromagnetic field one has $c_i^+ \rightarrow c_i^+e^{i\mathbf{F} \cdot \mathbf{r}}$. One must also assume that $H_{\text{int}}$ involves only density-density (charge or spin) interactions, so that it is trivially gauge invariant. This assumption has been questioned by Hirsch [11] who argues that 'occupation modulated hopping' terms of the form $c_i^+c_j(n_i + n_j)$ are important. These terms have not so far been much studied by other workers.

The Peierls assumption implies that the only important optical processes are those which move an electron from one site in the crystal to another. If there are optically allowed transitions between orbitals on the same site (e.g. the familiar hydrogenic $1s \rightarrow 2p$ transition), these must be added separately.

Once an $A$ dependence is determined it seems plausible that the current may be computed from the definition $j_r = \frac{i e}{N V_{\text{cell}}} \delta H / \delta A_r$ (here $V_{\text{cell}}$ is the volume of a unit cell, $N$ is the number of unit cells in the crystal and $r$ is a Cartesian coordinate). The Peierls coupling, for example, yields

$$j_r = -\sum_{i, \delta} \left[ i e^{ab}(\delta) (e^{i\mathbf{A} \cdot \mathbf{r}} c_i^+ c_{i+\delta}^+ c_{i+\delta} - e^{-i\mathbf{A} \cdot \mathbf{r}} c_i^+ c_{i+\delta}^+ c_{i+\delta}) \delta_r \right]$$

We emphasize the advantage of this procedure: if, as occurs for example in the Peierls substitution case, the $A$ dependence of the effective model may be found from general arguments, one obtains optical matrix elements without explicit computation of wave function overlaps (of course these are implicitly included in $H$; for example in the Peierls case through the overlap matrix $t^{ab}(\delta)$).
Application to effective models of the arguments justifying the f-sum-rule Eq (1.19) yields a restricted sum rule, which relates the oscillator strength in the conductivity of the effective model to the expectation value of an operator given by the part of \( j \) linear in \( A \). The form of the operator depends on the model; in the particular case of the Peierls coupled tight binding model the operator is the hopping amplitude weighted by the distance hopped and one finds

\[
\frac{2}{\pi e^2} \int_0^{\infty} d\omega \sigma_{r,s}^{th}(\omega) = \frac{1}{V_{cell}} \sum_{i,j} (t^{ab}(\delta) \delta_r \delta_s (c_{i,a}^+ c_{i+j,\delta,b}^+)) + H.c. \]  

(1.22)

Here \( r,s \) denote cartesian directions, \( \sigma^{th} \) denotes that portion of the conductivity arising from transitions among orbitals described by Eq (1.20) and again the sum rule applies both to infinite and finite lattices and to arbitrary boundary conditions. The restricted sum rule was apparently first noted by Maldague [12] and was rediscovered and its importance stressed by Baarswyl et al [13]; the present derivation comes from [14].

The restricted sum rule relates the spectral weight in a subset of all optical transitions (those described by the low energy effective model) to an expectation value, which may depend on many parameters including temperature and interaction strength. This should be contrasted to the full f-sum-rule which relates the integral over all optical transitions to the total carrier density and bare mass, which are temperature and interaction independent. The difference between the full and restricted sum rules is made up by transitions involving orbitals not included in the effective model. In particular, temperature and interaction strength dependence of the restricted sum rule is compensated by transitions between orbitals included in the effective model and orbitals not included in the effective model. At present there is no understanding of the relevant orbitals or energy range over which the full f-sum-rule is restored.

We now specialize to tight binding models and make the further assumptions that the lattice has orthorhombic symmetry with three lattice constants \( a_{1,2,3} \) and that the only appreciable hopping is between nearest neighbor sites. Then \( \sigma_{r,s} \sim \sigma_r \delta_{r,s} \) and

\[
\frac{2}{\pi e^2} \int_0^{\infty} d\omega \sigma_r^{th}(\omega) = \frac{a_1^2}{a_2 a_3} \sum_{i,r} (t^{ab}(\tilde{r}) \langle c_{i,a}^+ c_{i+r,\delta,b}^+ \rangle) + H.c. \equiv \frac{a_1^2}{a_2 a_3} K_r 
\]

(1.23)

In other words, if the hopping is only nearest neighbor a measurement of the optical conductivity gives the expectation value of the hopping amplitude, i.e. the electron kinetic energy \( K \). (Note that an optical experiment moves electrons in one direction only; thus it yields the kinetic...
energy of motion in that direction). Thus if the assumptions leading to Eq (1.23) apply, then an optical measurement yields a fundamental parameter of a many-body Hamiltonian, namely the expectation value of the kinetic energy. This is important because in the non-interacting limit, \( E = -K \) so the ground state wave function is the one which extremizes \( K \). If interactions are fundamentally important (i.e. if the ground state wave function is fundamentally different from the band theory one) then \( K \) will be substantially reduced from its non-interacting value. An explicit example of this phenomenon will be found below in the section concerning the Hubbard model.

On the other hand, for electrons interacting with phonons the situation is different. The standard theory of electron-phonon interactions involves two parameters: a dimensionless coupling, conventionally denoted \( \lambda \) and the ratio of a typical phonon frequency \( \omega_D \) to a typical electronic energy \( t \). In most cases \( \omega_D/t << 1 \), and the expansion parameter describing fluctuations about the ground state is \( \lambda \omega_D/t \). Provided \( \lambda \) is less than a (model dependent) critical value \( \lambda_c \), the ground state is essentially that given by band theory, the physics is described by the familiar Migdal-Eliashberg theory and one finds among other things that \( K = K_{\text{band}} - O(\lambda \omega_D/t) \). For \( \lambda > \lambda_c \) the ground state is fundamentally reconstructed (typically to an insulating 'polaronic' state) and \( K \) drops rapidly. For an explicit example see [15].

### 2.2.3 Down-folding and optical matrix elements.

The procedure of obtaining optical matrix elements via knowledge of the \( A \) dependence of an effective Hamiltonian is appealing and is important in practical terms. A more careful derivation is therefore desirable. Consider a formal 'down-folding' procedure: separate the Hilbert space into a low energy sector (\( L \)) involving orbitals of interest and a high energy (\( H \)) sector which we do not wish to explicitly consider. The Schroedinger equation \( H \psi = E \psi \) may be written in obvious notation as

\[
\begin{pmatrix}
H_H & H_M \\
H_M^* & H_L
\end{pmatrix}
\begin{pmatrix}
\psi_H \\
\psi_L
\end{pmatrix}
= E\begin{pmatrix}
\psi_H \\
\psi_L
\end{pmatrix}
\tag{1.24}
\]

The high energy subspace may be formally eliminated, leading to an equation involving only the low energy subspace which is

\[
H_{\text{eff}}(E)\psi_L \equiv [H_L + H_M^*(E - H_H)^{-1}H_M] \psi_L = E\psi_L
\tag{1.25}
\]

If we choose \( \langle \psi_L | \psi_L \rangle = 1 \) the exact (normalized) eigenfunction is

\[
|\psi\rangle = \frac{[(E - H_H)^{-1}H_M|\psi_L\rangle + |\psi_L\rangle]}{\sqrt{1 + \langle \psi_L | H_M^*(E - H_H)^{-1}H_M |\psi_L\rangle}}
\tag{1.26}
\]
Using the same splitting one can write the exact current operator in a form which depends explicitly only on the components of the wave function in the $L$ sector. If $N_1$ and $N_2$ are the (energy-dependent) normalization factors for the exact wave-functions corresponding to $\psi_{L1,2}$ then one finds

$$
\hat{j}_{L2} = [\langle \psi_{L1} | H^*_M ( E_1 - H)$

$$
+ \langle \psi_{L1} | H^*_M ( E_1 - H)^{-1} J_M | \psi_{L2} \rangle

+ \langle \psi_{L1} | J^*_M ( E_2 - H)^{-1} H_M | \psi_{L2} \rangle

+ \langle \psi_{L1} | j_{L} | \psi_{L2} \rangle ]/N_1 N_2 \quad (1.27)

Comparison of Eqs (1.25), (1.27) shows that the diagonal matrix elements of $j$ are correctly given by the $\langle \psi_{L1} | j | \psi_{L1} \rangle = \langle \psi_{L1} | \delta H_{eff}(E)/\delta A | \psi_{L1} \rangle$ (note the presence of $E$ in the normalization!). Off diagonal matrix elements within the $L$ subspace are not in general simply related to functional derivatives of $H_{eff}$ because of the two energies occurring in the matrix element. If in the energy range of interest one may neglect the variation of the operators above with $E$ then the matrix elements are correctly given by functional derivatives of $H_{eff}$ with respect to $A$. In the same way neglect of the variation of $H_{eff}$ with $E$ allows one to derive a restricted sum rule as above.

2.2.4 Accuracy of Peierls Phase Approximation. As mentioned above, a particularly convenient and widely-used effective model involves a tight-binding parametrization of band theory, along with some interactions. It is therefore important to consider how well the $A$—dependence of $H_{eff}$ is approximated by the Peierls phase ansatz. One issue concerns the importance of on-site ('1S-2P' like) transitions. This obviously depends on the system in question and cannot be discussed in general. Another concerns the possible relevance of the occupation modulated hopping terms in the interaction. Too little is known to allow discussion here. Even if these issues are neglected, a crucial question arises, related to the fact (seen e.g. in Eq (1.26)) that the physical wave function represented by the operator $c^+_i$ which we like to think of as creating an electron in an 'atomic-like' state of wave function $\phi_i$ in fact has a non-negligible fraction of its charge density coming from other orbitals, and may not be particularly well localized, so that the Peierls ansatz does not accurately describe the change in the wave function in the presence of a vector potential. There is also a nontrivial choice of basis aspect. To understand this, suppose the Hamiltonian, Eq. (1.20) is accepted (including the Peierls phase coupling to $A$). One may then change the basis from the original one $\phi_i$ to $\psi_i = \sum_j D(i-j) \phi_j$. In
the new basis \( t \) changes to \( t_{i-j}e^{iA(R_i-R_j)} \rightarrow t'_{i-j}(A) = \sum_{mn} D^*(i-m)t_{m-n}e^{iA(R_m-R_n)}D(n-j) \neq t'_{i-j}e^{iA(R_i-R_j)}. \) Thus, the Peierls phase ansatz can at most be correct in one basis, and does not have to be correct in any basis.

There is presently substantial interest in transition metal oxides in which the important electronic states are relatively narrow bands derived from transition metal d-states (hybridized with oxygen p-states) and are relatively isolated from other bands. For such systems I believe that the 'correct' basis choice for \( t_{i-j} \) is the one in which the Peierls phase ansatz most nearly approximates diagonal matrix elements (in particular those giving rise to the charge stiffness) computed via other techniques. For example, it will be seen below that within band theory there is a standard expression for the charge stiffness, which one may compare to that obtained from the Peierls ansatz in a given tight binding basis.

The accuracy of the Peierls-phase matrix elements has not been the subject of systematic study in realistic situations. Ahn and Millis [16] have investigated the Kronig-Penney model of electrons in one spatial dimension in the presence of a periodic array of delta functions. They compared the exact conductivity to the result obtained by making a nearest neighbor tight-binding fit to the lowest-lying bands and found that in all reasonable situations even the oversimplified nearest neighbor approximation yielded spectral weights accurate to within 10%. Some of their results are reproduced as Fig. (1.2).

2.3 Simple Examples

2.3.1 Galilean Invariant Models. A situation frequently encountered in textbooks and occasionally in practice is the Galilean-
invariant limit. If no ions are present (or the spatially varying part of the ionic potential can be neglected, as is the case for a low density of electrons in a clean semiconductor), and if (as is usually the case) $H_{el-el}$ is a function of relative positions of electrons only, then the current is proportional to the momentum and is conserved: $[H, \vec{j}] = 0$. In this circumstance $\langle n|j_a|m \rangle = 0$ if $n \neq m$. Further, states $|n\rangle$ excited with non-negligible thermal probability $e^{-\beta E_n}/Z$ have a current (expectation value of $j$) which vanishes in the infinite system size limit. Thus at least in the infinite system size limit the current-current correlation term $\chi$ can usually be neglected and one has (GI stands for Galilean-invariant)

$$\sigma^{GI}_{ab}(\omega) = \left( \frac{ne^2}{im(\omega - i\varepsilon)} \right) \delta_{ab}$$ (1.28)

In summary, in a Galilean-invariant model (with arbitrary but Galilean-invariant interactions) the real part of the conductivity is entirely concentrated at $\omega = 0$: the response to a non-zero-frequency field is entirely reactive; the carriers are simply freely accelerated by the electric field.

2.3.2 Band Theory. Modern band theory is not a non-interacting theory. Electron-electron interactions are taken into account by different approximations to density functional theory [17]. The implementation most widely used is based on an effective single-particle Schrödinger equation involving a non-local 'exchange-correlation potential' which contains a significant contribution from electron-electron interactions and is determined by a self consistency condition. Solving this equation yields a set of effective one-electron energy levels $\varepsilon_n^{\text{eff}}(p)$ and wavefunctions $\psi_{n,p}(r)$ such that ($\mu$ is the chemical potential) $E_{\text{DFT}} = \sum_n \int \frac{d^3p}{(2\pi)^3} \varepsilon_n^{\text{eff}}(p) \Theta(\mu - \varepsilon_n^{\text{eff}}(p))$ is a good approximation to the ground state energy (the approximation would be exact if the exact non-local potential were used and if the equation could be solved exactly, and the energetics obtained from standard approximations are often remarkably good in practice). The $\varepsilon_n^{\text{eff}}(p)$ and $\psi_{n,p}(r)$ themselves have in principle no rigorous meaning, but are often interpreted as actual electron energy levels and wave functions. Within band theory for a static, perfectly ordered lattice one expects that the real part of the conductivity has a delta function contribution from states at the fermi surface (if the material is predicted to be a metal) and interband contributions generically separated from $\omega = 0$ by an energy gap, so $\sigma^{\text{re}}(\omega) = 0$ for a non-zero range of frequencies around $\omega = 0$ [8].

The arguments leading to Eq (1.13) may be applied to the density functional formalism to obtain an expression for $D$. This expression
would be exact if the flux dependence of the exact exchange-correlation potential were known. In the band theory literature, the flux dependence of the exchange correlation potential is neglected, so the flux only enters in the derivative term of the Schroedinger equation. \( D \) may then be computed easily and is [8]:

\[
D_{\text{band,ab}} = \sum_n \int \frac{d^3p}{(2\pi)^3} \frac{\partial \varepsilon_n^{\text{eff}}(p)}{\partial p_a} \frac{\partial \varepsilon_n^{\text{eff}}(p)}{\partial p_b} \delta(\varepsilon_n^{\text{eff}}(p) - \mu)
\]  

(1.29)

In other words, if the flux dependence of the exchange-correlation potential is neglected then \( D \) is given by the average over the fermi surface of the appropriate components of the fermi velocity multiplied by the density of states. For many correlated materials, this is a very poor approximation to \( D \), and the discussion of fermi liquid theory in the next subsection shows that neglect of the flux dependence of the exchange correlation potential is in principle incorrect.

An extensive literature exists on optical properties computed using Eq (1.6) with band theory wave functions used to compute the matrix elements, and with additional interaction corrections added using various roughly speaking perturbative extensions of band theory (most notably the 'GW' approximation); for a review see, e.g. [18]. Recent important work has addressed electron-hole correlation effects [19]. This approach involves extremely heavy computations, and has not yet been widely applied to 'strongly correlated' transition metal oxides (but see [20]).

2.3.3 Integrable Systems. One expects on general grounds that at \( T > 0 \), \( D = 0 \): even in the absence of disorder, many-body interactions will broaden the delta function in some manner. However, theoretical work over the last decade [6, 21] strongly suggests that in many integrable one dimensional models, \( D > 0 \) at all \( T \), so the systems are infinitely conducting at all temperatures. This peculiar result is apparently a consequence of the infinite number of conservation laws characteristic of integrable systems. The issue is discussed in more detail in Chapter 11.

2.3.4 Fermi Liquid Theory. The low energy properties of many interacting electron systems are believed to be well described by L. D. Landau's 'fermi liquid theory' (for references, see e.g. [3, 22]). One crucial property of a fermi liquid is the existence of electronic 'quasiparticles' which behave in many ways as conventional electrons, but with a renormalized dispersion \( v^*(p) = \frac{m_{\text{band}}(p)}{m^*(p)} \frac{\partial \varepsilon_n^{\text{eff}}(p)}{\partial p_a} \) where the momentum label indicates position on the fermi surface and we have defined a
'mass enhancement' $m^* / m_{\text{band}}$. Response functions of fermi liquids are determined by a combination of quasiparticle dispersion and 'Landau parameters' representing the feedback of the system on a given electronic state. Fermi liquid theory leads to

$$D_{\text{observed}} = D_{qp}(1 + F_{1s}/d) \quad (1.30)$$

where $D_{qp}$ is obtained by using the quasiparticle dispersion in Eq (1.29), $d$ is the dimensionality and $F_{1s}$ is a Landau parameter. In a Galilean-invariant system the mass renormalization is independent of position on the (spherical) fermi surface and the relation $m^*/m = 1 + F_{1s}/d$ which follows from Galilean invariance ensures that $D$ is unrenormalized. In a non-galilean-invariant system there is no such relation. If the many-body renormalizations involve a strictly momentum independent self energy then $F_{1s} = 0$ [23] and in 'heavy fermion' materials the effects due to velocity renormalizations are much larger (factor $\sim 100$) than those due to $F$, although the Landau parameter effects have been measured in a few cases and are important at the factor of two level [24]. On the other hand, in a fermi liquid near a ferromagnetic transition the effective mass diverges as the transition is approached, but because the critical modes involve long wavelength fluctuations (i.e. mainly forward scattering) one does not expect the conductivity to be strongly affected. Thus in this case the Landau parameter must diverge along with the effective mass.

2.3.5 Superconductivity and Density Waves. As the temperature is decreased a fermi liquid may become unstable to various forms of long ranged order. Two particularly instructive special cases are the superconducting and density wave instabilities.

A superconductor is characterized by a non-vanishing superfluid stiffness $\rho_s = \frac{1}{4e} \lim_{q \rightarrow 0, \omega \rightarrow 0} (\omega \sigma(q, \omega))$ where the $1/4$ is conventional and refers to the charge $2e$ of a Cooper pair. The magnetic field penetration depth may be inferred from the $q \rightarrow 0, \omega = 0$ limit of the response, and in superconductors the order of limits does not matter, so that $\sigma(\omega)$ has a term proportional to $\delta(\omega)$ whose magnitude may be inferred from the penetration depth [25]. A superconductor is also characterized by an energy gap $\Delta$ which in conventional superconductors is much less than the characteristic conduction band energy scale $E_F$. In the conventional theory, up to terms of relative order $(\Delta/E_F)^2$ the total spectral weight in a superconductor is the same as that of the corresponding normal system, so the weight in the superfluid stiffness is mainly transferred down from higher frequencies (within the conventional theory it comes from $\omega$ of the order of a few times $\Delta$).
These arguments were introduced and experimentally verified by Tinkham in the late 1950s and are beautifully explained in [25]. In high temperature superconductors, conservation of spectral weight as temperature is varied across the superconducting transition was verified at the 10% level in the early 1990s [26]. Of course, small changes in conduction band spectral weight are expected as the temperature is varied through the superconducting transition temperature $T_c$, and recent improvements in experimental technique have allowed these changes (which seem to be at the 1% level) to be observed [27] (see Chapter 7 for further discussion of this issue).

A density wave occurs when the electron charge or spin density acquires a periodicity different from that of the underlying lattice. One expects this density modulation to cause an additional periodic potential (energy gap $\Delta$) which is felt by the mobile electrons, which eliminates some of the Fermi surface. Just as in superconductors the total weight is expected to be conserved (up to terms of relative order $(\Delta/E_F)^2$) so the formation of the density wave is expected to shift spectral weight up in frequency. This behavior is observed in many density wave materials; an example is shown in the upper panel of Fig 1.3 [28]. Remarkably, in some materials (most notably the dicalcogenides such as $TaSe_2$) the expected upward shift does not occur [29]: instead, as the temperature is lowered across the density wave transition spectral weight shifts downwards in frequency, apparently because the scattering...
rate is reduced. This behavior is shown in the lower panel of Fig (1.3). An understanding of the different origins of these two behaviors would be very desirable.

3. Specific Model Calculations

3.1 Direct evaluation of Kubo formula

3.1.1 Formalism: momentum independent self energy.

In this approach one starts from a model of electrons with a given dispersion $\varepsilon_p$ (typically interband transitions are neglected), a coupling to the electromagnetic field given by $p \rightarrow p - eA/c$ and interactions which are treated by the methods of diagrammatic perturbation theory. Important objects in these calculations are the electron propagator $G$ and self energy $\Sigma$:

$$G(p, \omega) = \frac{1}{\omega - \varepsilon_p - \Sigma(p, \omega)} \quad (1.31)$$

A general expression for the conductivity is [30]

$$\sigma_{ab}(i\Omega) = \sigma^{dia}_{ab} + \sigma^{para}_{ab} \quad (1.32)$$

with

$$\sigma^{dia}_{ab}(i\Omega) = \frac{2e^2}{\Omega} \sum_\omega \int \frac{d^3p}{(2\pi)^3} \frac{\partial^2 \varepsilon_p}{\partial p_a \partial p_b} G(p, \omega) \quad (1.33)$$

and

$$\sigma^{para}_{ab}(i\Omega) = -\frac{2e^2}{\Omega} \sum_\omega \int \frac{d^3p}{(2\pi)^3} \frac{\partial \varepsilon_p}{\partial p_a} G(p, \Omega + \omega) G(p, \omega) T^{\Omega}_{b}(p, \omega) \quad (1.34)$$

The vertex function $T_b$ satisfies the integral equation

$$T^{\Omega}_{b}(p, \omega) = \frac{\partial \varepsilon_p}{\partial p_b} - 2T \sum_{\omega'} \int \frac{d^3p'}{(2\pi)^3} I^{\Omega}_{p'}(\omega, \omega') G(p', \Omega + \omega') G(p, \omega') T^{\Omega}_{b}(p', \omega') \quad (1.35)$$

and $I$ is a particle-hole irreducible vertex, whose limit as $\Omega, \omega \rightarrow 0$ and $p \rightarrow p_F$ is the Landau interaction function. In general an expression for $I$ is difficult to determine and the equation is difficult to solve: quantum Boltzmann equation methods [31] have been more useful in practice.

One instructive special case which can be analysed in detail is a momentum independent self energy, $\Sigma(p, \omega) \rightarrow \Sigma(\omega)$. This situation is (to a good approximation) realized in practice for a high density of electrons coupled to phonons [32], in the 'dynamical mean field' or 'd = \infty'...
approximation [23] and in the 'marginal fermi liquid' model for high temperature superconductivity [33]. In these cases, the vertex correction vanishes and for a non-superconducting system on the imaginary frequency axis (the $2$ is for spin and $\omega_+ = \omega + \Omega$)

$$
\sigma_{ab}(i\Omega) = \frac{2\epsilon^2}{i\Omega} \sum_{\omega} \int \frac{d^3p}{(2\pi)^3} \left[ \frac{\partial^2 \epsilon_p}{\partial p_a \partial p_b} G(p, \omega) - \frac{\partial \epsilon_p}{\partial p_a} \frac{\partial \epsilon_p}{\partial p_b} G(p, \omega_+) G(p, \omega) \right]
$$

(1.36)

A note on units: for simplicity (and because it is the case of greatest experimental relevance) we consider an orthorhombic lattice with lattice constants $d_{a,b,c}$. Making the momentum integrals dimensionless via $p'_a = d_{a,b} p_a$ etc and integrating by parts on the first term (recall $\partial \Sigma/\partial p$ is assumed to vanish) leads to

$$
\sigma_{ab}(i\Omega) = \frac{2\sigma_0}{i\Omega} \sum_{\omega} \int \frac{d^3p'}{(2\pi)^3} \frac{\partial \epsilon_p}{\partial p'_a} \frac{\partial \epsilon_p}{\partial p'_b} \left[ G(p', \Omega + \omega) G(p', \omega) - G(p', \omega)^2 \right]
$$

(1.37)

where

$$
\sigma_0 = \frac{e^2}{\hbar c}
$$

(1.38)

The conductivity evidently has the dimension of energy and may be converted to conventional units by recalling that $e^2/\hbar = 4k\Omega$.

Eq (1.37) may be easily evaluated numerically if $\Sigma(\omega)$ is known. A widely studied limit arises if, for all relevant frequencies, $\Sigma$ is small compared to the regime over which $\epsilon_p$ varies. To be precise, if $-W_1 < \epsilon_p < W_2$ then up to terms of order $\Sigma/(\min(W_1, W_2))$ one may use a pole approximation to perform the integral over the magnitude of $\epsilon_p$. Performing the standard analytical continuation leads for $\Omega > 0$ to

$$
\frac{\sigma_{ab}(i\Omega)}{\sigma_0} = \int \frac{d\omega d\cos(\theta) d\phi}{2\pi} \frac{\nu_a \nu_b \nu_0(\theta, \phi) (f(\omega) - f(\omega_+)) / \Omega}{\Omega - \Sigma'(\omega_+) + \Sigma'(\omega) - i (\Sigma''(\omega_+) + \Sigma''(\omega))}
$$

(1.39)

Here $\omega_+ = \omega + \Omega$, $\nu_{a,b}$ are the $a, b$ components of the fermi velocity, $f$ is the fermi function and $\nu_0(\theta, \phi) = 1 / |\partial \epsilon_p / \partial p|$ at the fermi surface point specified by the angles $\theta, \phi$.

3.1.2 Electron-phonon interaction. For electrons interacting with dispersionless optical phonons of frequency $\omega_0$ the self energy
Figure 1.4. Optical conductivity, 'optical scattering rate' and single-particle scattering rate for model of electrons coupled to dispersionless optical phonons, computed from Eqs (1.39, 1.40, 1.41, 1.16). Left panel (a): high temperatures (\( T = \Omega_D / 2 \)) Right panel (b): low temperature (\( T = 0.1 \Omega_D \)). Frequencies measured in units of \( \Omega_D \). Coupling \( \lambda = 1 \). Units of scattering rate and conductivity are arbitrary; frequency is scaled to phonon frequency. The scales for the optical and single particle rates are identical.

on the real frequency axis is [22] (\( f \) is the fermi distribution function)

\[
\Sigma'_{ep}(\omega) = \lambda \int d\omega f(\omega') - \omega_0^2 \omega_0^2 - \omega' \omega''
\]

\[
\Sigma''_{ep}(\omega) = \frac{\pi \lambda \omega_0}{2} \left( \coth \left[ \frac{\omega_0}{2T} \right] - \frac{\sinh \left[ \frac{\omega_0}{2T} \right]}{\cosh \left[ \frac{\omega_0}{2T} \right] + \cosh \left[ \frac{\omega_0}{2T} \right]} \right)
\]

The panels of Fig (1.4) show the real part of optical conductivity, the optical and the single-particle scattering rate computed from Eq (1.39) using Eqs (1.40,1.41). The computation was performed for two different temperatures—one much lower than the Debye frequency and one equal to \( \omega_D / 2 \). For \( T = \omega_D / 2 \) (panel a) the conductivity is already not very far from the simple 'Drude' form, as seen from the conductivity and from the
frequency dependence of the scattering rate. For the lower temperature (panel b), one sees a large low frequency peak (very weakly scattered electrons, with a mass increased by the electron-phonon interaction) and an extra absorption beginning at the phonon frequency. One sees that the 'optical mass' and scattering rate are roughly speaking 'smoothed' versions of the single-particle mass and scattering rate.

3.1.3 Marginal fermi liquid ansatz. The marginal fermi liquid ansatz [33] is a theoretical prediction for the electron self energy of optimally doped high-$T_c$ superconductors, which seems to have been borne out by recent photoemission experiments [34]. It has not been convincingly derived from any microscopic model. The marginal fermi liquid ansatz for the imaginary part of the electron self energy is that at frequencies $\omega$ less than a cutoff frequency $\omega_c$ one has

$$\Sigma''(\omega) = \pi \lambda \max(\omega, \pi T)$$  \hspace{1cm} (1.42)

If one assumes (as is normally done in the literature) that for frequencies greater than $\omega_c$, $\Sigma''(\omega) = \lambda \omega_c$ then

$$\Sigma''_{MFL}(\omega) = \lambda \omega_c \left( \ln \frac{1 - \frac{\omega}{\omega_c}}{1 + \frac{\omega}{\omega_c}} - \frac{\omega}{\omega_c} \ln \left| \frac{\frac{\omega^2}{\omega_c^2} - 1}{\frac{\pi T}{\omega_c} - 1} \right| \right) - \frac{\pi T}{\omega_c} \ln \left| \frac{\pi T - \omega}{\pi T + \omega} \right|$$  \hspace{1cm} (1.43)

This self energy has been used with Eq (1.37) to analyse the frequency dependent conductivity of high-$T_c$ superconductors; for results see [35]; but Drew and the author have presented evidence that Landau parameter effects are also important, at least at the factor-of-two level [36].

3.1.4 Weakly coupled lower-dimensional subsystems. An important sub-class of conductivities involves the motion of charge between weakly coupled lower dimensional subsystems; for example the interplane conductivity of high temperature superconductors or the interchain conductivity of quasi-one-dimensional materials. This situation may be described by a Hamiltonian of the form

$$H = -t_\perp \sum_{i,p,\sigma} \left( c_{i,p,\sigma}^+ c_{i+1,p,\sigma} + H.c. \right) + H_|| + H_{inter}$$  \hspace{1cm} (1.44)

where the label $p$ denotes the momentum in the lower dimensional sub-system (plane or chain), $\sigma$ is spin, and we have labelled the different planes (or chains) by $i$. Here $H_||$ is the Hamiltonian describing the physics in an isolated low dimensional subsystem and $H_{inter}$ denotes any interplane (interchain) interactions. In practice the only important
contribution to $H_{\text{inter}}$ is likely to be the coulomb interaction. We refer to the direction(s) in which the hybridization is weak as the 'transverse' directions.

The transverse conductivity is typically computed by using the Peierls ansatz to couple the electromagnetic field and then expanding as usual. The crucial new point is that if $t_\perp$ is weak compared to some appropriate in-plane or in-chain energy scale, the conductivity may be computed by perturbation theory in $t_\perp$. At leading nontrivial order one has

$$
\sigma_\perp = \frac{t_\perp}{i\Omega} \langle c_{i,p,\sigma}^+ c_{i+1,p,\sigma} + H.c. \rangle + \frac{\chi_\perp(i\Omega)}{i\Omega}
$$

(1.45)

with $\chi$ given in the time domain by the usual commutator

$$
\chi_\perp = \sum_{p,\sigma} \langle \{ j_\sigma(t), j_\sigma(0) \} \rangle
$$

(1.46)

with

$$
j_\sigma(t) = it_\perp \left( c_{i,p,\sigma}^+ c_{i+1,p,\sigma} - H.c. \right)
$$

(1.47)

Here the momentum sum is over in-plane or in-chain momenta only. To leading order in $t_\perp$ the expectation value which defines $\chi_\perp$ may be calculated assuming $t_\perp = 0$ while to obtain the first term one must calculate to first order in $t_\perp$. If the term $H_{\text{inter}}$ may be neglected (or treated in mean field theory) then the results may be simply expressed as products of Green functions pertaining to $H_\parallel$. In the absence of superconductivity or density wave order, the only nonvanishing Green function is $G(p,t) = \langle T\{ c_{p\sigma}(t), c_{p\sigma}(0) \} \rangle$ and

$$
\sigma_\perp = T \sum_{\omega,p,\sigma} \frac{G(p,\omega + \Omega)G(p,\omega) - G^2(p,\omega)}{i\Omega}
$$

(1.48)

This formula is useful because in many cases the in-plane (in-chain) Green function is known, so the conductivity may be directly computed. For applications to the interplane conductivity of 'single-layer' high temperature superconductors see [37]; to bilayer cuprates see [38]; to quasi one dimensional materials see [39]. Eq (1.48) is however only the leading term in a perturbative expansion in $t_\perp$. Essler and Tsvelik have recently noted that for a particular form of $t_\perp$ a controlled treatment of the $1d$-$2d$ crossover (including optics) may be constructed [40].

The applicability of this formula has been questioned by Turlakov and Leggett, who argue that 'Coulomb blockade' effects similar to those producing tunnelling anomalies in disordered systems may be important [41]. The issue deserves further analysis.
3.2 The Hubbard Model

The Hubbard model is defined by the Hamiltonian

\[ H_{\text{hub}} = - \sum_{i,j} t_{i,j} (c_{i\uparrow}^* c_{j\uparrow} + H.c) + U \sum_i n_{i\uparrow} n_{i\downarrow} \tag{1.49} \]

Most studies have assumed a d-dimensional cubic lattice and a hopping which is nonvanishing only between nearest neighbors. The Hubbard model displays a Mott transition: at a density of one electron per site and for a large enough interaction the ground state is insulating, \((D = 0)\) and characterized by a gap to charge excitations. The main interest has been in the behavior of the conductivity (and other physical properties) in the vicinity of the Mott insulating phase. It is conventional to describe the carrier density in terms of a doping, \(\delta = 1 - n\) away from half filling.

Consider now the kinetic energy and optical properties. For \(n\) far from 1, even very strong interactions have a weak effect on \(K\) essentially because the electrons can avoid each other; also the model is nearly galilean-invariant. For \(n\) near 1 a large \(U\) can have profound effects. For a density of one electron per site and sufficiently large \(U\) the ground state is insulating \((D = 0)\) although band theory would predict it to be metallic. At \(n = 1\) and very large \(U\) a good approximation to the ground state is one with one electron per site and an insulating gap \(\Delta_{\text{opt}} = U - \alpha_d t\) and \(a_d\) a dimension-dependent constant.

Hopping leads to fluctuations into states with two electrons or no electrons per site and thus if \(t < U\) to \(K \sim t^2/U\). As \(t/U\) decreases, \(K\) increases, eventually saturating at the band theory value. Depending on the details of the band structure, the insulating behavior may persist down to arbitrarily small \(U\) (at \(n = 1\)) or there may be a critical \(U\) at which a Mott transition occurs. Whether or not this happens, one may distinguish large and small \(U\) by whether the kinetic energy is substantially (factor of 2) renormalized from the band theory value or not. In essentially all models, the large \(U\) regime extends down to \(U \approx 2dK\) where \(d\) is the spatial dimensionality (recall \(K\) was defined as the hopping in one cartesian direction). For large \(U\) and \(n\) near but not equal to unity there is a small density of holes (if \(n < 1\)) or doubly occupied sites (if \(n > 1\)) and these can move more or less freely, leading to \(K \sim t^2/U + |1 - n| t\). For \(|1 - n|\) not small, the carriers mostly avoid each other; the renormalization of the kinetic energy is small and the state is more or less conventional.
In the $d = 1$ Hubbard model an exact solution is available. The zero temperature kinetic energy and Drude weight were computed by Schulz [42] and later studied in more detail by other workers [43]. Representative results are shown in Fig. 1.5. Note that as the Mott phase is approached by varying doping, $(\delta \to 0)$ $D$ vanishes linearly in $\delta$, with a coefficient which depends on interaction strength, and which may be interpreted as a correlation length $\xi$ (normalized to the lattice constant) for the Mott insulating phase [43].

For not too large $U$ (i.e. $\xi > 1$) the numerical results may be written as $D(\delta, U) = D_0 f(\delta \xi)$ with $f$ a scaling function discussed in detail in [43]. For $\delta \xi \gtrsim 0.4$, $f \to 1$, implying among other things that for $\delta \xi > 1$ the spectral weight rapidly collapses into the Drude peak. This special feature of one dimensional kinematics occurs even at larger $U$ where the scaling theory does not apply, and is discussed in e.g. [44]. Note also that to observe a significant suppression of total spectral weight one requires a $U$ at least of order the full band-width, and a $\delta \lesssim 0.4$.

For higher dimensions few (see for example [14]) exact results are available. Numerical studies of small clusters are available for $d = 2$ (essentially no useful results exist in $d = 3$); representative results [45] are shown in Fig. 1.6. Little is understood about finite size effects except in $d = 1$ where they have been extensively studied and found to be unpleasantly large [43]. The kinetic energy should be re-examined with the improved computers and algorithms available today.

If the momentum dependence of the self energy is negligible $(\Sigma(p, \omega) \to \Sigma(\omega))$ then a formally exact solution (which must still be implemented numerically) is available. This approximation, which is believed to be reasonably accurate in $d = 3$ is known as the 'dynamical mean field approximation' and has been used to compute the kinetic energy as a function of interaction and doping. Unpublished results obtained by Ferrara and Rozenberg for the kinetic energy are shown in the lower panel of Fig 1.6. Kotliar and collaborators have analysed the conductivity.
near the Mott transition using this approximation, and compared the results to data on V$_2$O$_3$ obtained by G. A. Thomas and co-workers [46]. Fig (1.7) presents theoretical results, along with a qualitative view of the theoretically expected structure, while Fig (1.8) shows the measurements.

The electron green function is characterized by three features: a lower Hubbard band, an upper Hubbard band, and a quasiparticle peak in between. The conductivity correspondingly has three peaks corresponding to motion within the quasiparticle band, transitions from the quasiparticle band to the upper Hubbard band (or from the lower Hubbard band to the quasiparticle band) and from the lower to the upper Hubbard band.

One fundamental question which, remarkably, has not yet been fully answered concerns the form of the conductivity in the Mott insulating state. The model has a gap to charge excitations, so one expects a conductivity with vanishing real part at low frequencies, and a conductivity onset for frequencies above the Mott gap energy $E_{\text{Mott}}$. Information should be obtainable in $d = \infty$, but the author is unaware of specific results. In $d = 1$ analytical and numerical renormalization group studies of Joekelmann, Gebhard and Essler [47] indicate that that

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.6.png}
\caption{Upper panel: Kinetic energy $K$ for Hubbard model in $d = 2$ for $U = 0, 4, 8, 20$; Lower panel: $K$ in dynamical mean field approximation for $U = 0, 1, 2, 4$. The different bandwidth conventions mean that the U-values are not directly comparable; a useful rule of thumb is that when $U$ equals the full bandwidth, $K(n = 1, U) \approx K(n = 1, U = 0)/2$}
\end{figure}
Figure 1.7. Calculated optical conductivity of V$_2$O$_5$ on insulating and metallic sides of Mott transition [46] The uppermost curves, labelled $\rho(\omega)$ show the changes in the electron spectral function as the metal is driven (by increasing interaction) from a correlated metal state (left graph) to a Mott insulating state (right graph). The lower graphs give the corresponding changes in conductivity.

\[ \lim_{\omega \to E_{\text{Mott}}} \sigma(\omega) \propto \sqrt{\omega - E_{\text{Mott}}} \text{ and that the maximum in } \sigma \text{ occurs at } \omega \approx 1.25E_{\text{Mott}}. \]

3.3 t-J Model

The Hubbard model is an approximation to the physics of transition metal oxides on energy scales of the order of the conduction band width (if this is relatively narrow and relatively well separated from other bands in the solid). A further approximation, valid for large $U$ and low energies, is the 't - J' model [48]. This may be derived from the Hubbard model by a formal canonical transformation procedure, and is of the form

\[ H_{tJ} = P_D \left[ -\sum_{i,j} t_{i,j} \left( c_{i\sigma}^+ c_{j\sigma} + H.c. \right) + J \sum_{<ij>} \vec{S}_i \cdot \vec{S}_j + ... \right] P_D \quad (1.50) \]

Here $P_D$ annihilates states containing doubly-occupied sites, $J \sim t^2/U$ and the ellipsis denotes terms of higher order in $t/U$.

As defined here the $t - J$ model is an effective model describing the low energy physics of models with strong on-site repulsion. Application of the standard f-sum-rule derivations shows that

\[ \frac{2}{\pi \epsilon^2} \int_0^\infty d\omega \sigma_{r,s}^{t-J}(\omega) = \frac{1}{V_{\text{cell}}} \sum_{i,\delta,\sigma} (t(\delta)\delta_r \delta_s (P_D c_{i\sigma}^+ c_{i+\delta,\sigma} P_D) + H.c.) \quad (1.51) \]
Figure 1.8. Measured optical conductivity of V$_2$O$_3$ on insulating and metallic sides of Mott transition, compared to 'dynamical mean field' calculations (from [46]).

The projectors imply that the total spectral weight in the transitions described by the $t - J$ model vanishes as $\delta \to 0$, as therefore, does the charge stiffness $D$.

Prevlosek has developed an interesting numerical technique to determine the frequency dependent conductivity of the $t - J$ model [49]. Other workers have not followed up on these methods.

3.4 Charge transfer insulators

The Hubbard model involves two energy scales, $t$ and $U$. It is only a useful representation of low energy physics if both $t$ and $U$ are small compared to the band gaps separating the orbitals of interest from other orbitals in the solid. In many presently interesting correlated systems (in particular transition metal oxides) this is not the case: the $U$ is so large that the basic charge transfer process involves shifting a carrier from an orbital on a transition metal site to an orbital on a different ion altogether (most commonly an oxygen ion). Such systems are referred to as 'charge transfer' rather than 'Mott-Hubbard' systems [50]. Charge transfer systems are in many respects similar to Mott-Hubbard systems: in particular, they display a correlation driven metal insulator transition at commensurate densities and for nearby dopings the low energy physics is believed to be described by the $t - J$ model [51]. The charge
stiffness and low frequency conductivity are therefore presumably similar to those of the t-J model. However, (and surprisingly, considering the experimental relevance) the form of the conductivity at larger frequencies (for example near the gap energy in the insulator) has not been investigated.

3.5 Kondo Lattice Model

3.5.1 Overview. A wide class of condensed matter phenomena involve carriers interacting with spins, and the basic model describing this situation is the Kondo lattice model:

$$H_{KL} = \sum_{\sigma} \int \frac{d^3 p}{(2\pi)^3} \varepsilon_p c_p^+ c_p + J \sum_{i} \vec{S}_i \cdot \vec{\sigma}_\alpha \sigma_{i\alpha}^+ \sigma_{i\beta}$$  \hspace{1cm} (1.52)

This model has two interesting limits: if the magnitude of the coupling $JS$ is large (relative to the electron band-width) then the carrier spin on site $i$ is 'slaved' to the spin of the local moment. This 'double exchange' (the term is historical) limit is apparently relevant to the 'colossal' magnetoresistance manganites and to a variety of related systems. On the other hand, if the spin magnitude $\vec{S}_i = 1/2$, the coupling is antiferromagnetic and the magnitude of $J$ is small, then it is possible for the Kondo effect to 'marry' the conduction electrons to the local moments, yielding a 'heavy fermi liquid'. We consider the two limits separately.

3.5.2 Double exchange. The most extensively studied 'double exchange' systems are the 'colossal magnetoresistance' manganese perovskites (and related Ruddlesden-Popper systems). For reviews see, e.g. [52]. The crucial physics here is a very large $J$ which arises from the atomic Hunds coupling. Its magnitude has not been measured directly. Quantum chemical considerations and experience with gas-phase $Mn$ and other $Mn$ compounds suggests [53] that the isolated-ion level splitting ($e_g$ parallel to $e_g$ antiparallel to core spin) $2J_{Hund}\Delta S_c$ is about 2.5eV. Optical experiments suggest that in the actual CMR materials it is somewhat larger; at least 4eV [54].

The strong coupling of carriers to spins leads to physics called "double exchange" which has very interesting consequences for a number of properties including optics. The essential point is this: the coupling between mobile electrons and core spins is apparently so strong that at physically relevant energies a mobile electron on a given site is constrained to have its spin parallel to the core spin on that site. This implies that the amplitude for an electron to hop from one site to another is modulated by a spin overlap factor which is maximal when the core spins on the two sites are parallel and is minimal when the two core spins are antiparallel.
Ferromagnetic alignment of the core spins increases the electron kinetic energy (and is indeed the driving force for ferromagnetic order in these compounds), antiferromagnetic alignment decreases it, and a random spin arrangement reduces it by a factor of approximately 1/√2. Over a wide range of parameters, 'double exchange' models specified by Eq (1.52) (perhaps supplemented by other interactions) have ferromagnetic ground states with a Curie temperatures relatively small in comparison to the electronic band-width [52]. Thus by varying the temperature over a range small compared to the band-width a large change in the kinetic energy can be seen.

Figure 1.9 shows one example of this phenomenon: results of theoretical calculations (performed using the 'dynamical mean field approximation' and a simplified band structure) of the optical conductivity of a double exchange model with \( J_H \to \infty \) and an additional electron-phonon interaction [15]. A strong temperature dependence of the functional form and integrated area is evident.

Figure (1.10) shows the temperature dependence of the kinetic energy (which may be directly computed and for this model is equal to the integrated area under the conductivity). The Curie temperature is evident as a kink in the curve. In the double-exchange-only model \( (\lambda = 0) \) the approximately 1/√2 change in \( K \) between \( T_c \) and \( T = 0 \) is seen. For increased interactions, a larger change occurs, because the change in the effective \( t \) changes the balance between the kinetic energy and the
other interaction, and thus the expectation value of the hopping operator. However, for this class of models the largest change which occurs is about 50% of the \( T = 0 \) kinetic energy. Finally, one can see that for this model at least, the ratio \( T_c/K(T_c) \approx 0.2 \), relatively independent of interaction strength. Extending this analysis to other interactions would be very important.

Note that the model studied in ref [15] involves classical phonons, so the 'Migdal parameter' is \( T \lambda \). The differences seen between the \( \lambda \)-dependence of \( K(T = 0) \) (negligible for \( \lambda < \lambda_c \)) and that of \( K(T > T_c) \) (non-negligible for \( \lambda T > 1 \)) constitute an explicit example of the effect of the electron-phonon interaction on the electron kinetic energy. In the very large J limit, the magnitude and temperature dependence of the observed spectral weight may be related to the magnetic transition temperature; for details see [54], but very recent work [55] indicates that there is no simple relationship for realistic J values.

3.5.3 Heavy fermions. In heavy fermion materials, the carrier-spin interaction leads (via a lattice version of the Kondo effect) at low temperatures to the formation of a 'heavy' fermi liquid characterized by a very long mean free path and a quasiparticle mass which may be as much as 50-100 times the band mass. For a review see e.g. [56]. In a number of cases the enhanced mass has been observed directly via quantum oscillation measurements [57]. The low temperature electronic properties may be described by a self energy which has a very strong frequency dependence and a much weaker (in many cases negligible) momentum dependence [58]. Substitution into Eq (1.39) suggests that the conductivity should exhibit a Drude peak with a very small amplitude (reduced from the band theory value by roughly the same \( \sim 100 \) factor as that by which the quasiparticle mass is increased) [59]. This has at
least qualitatively been observed [60]. Analyses of the temperature dependence of the superconducting penetration depth [24] suggests that Landau parameter effects are also important on the factor of two level, at least in some heavy fermion materials.

4. Application to Data–CMR and High Tc

4.1 High-Tc superconductors

The high-Tc superconductors are electronically two dimensional materials in which the basic unit is the "CuO2 plane". This has a basically square symmetry with lattice constant $a \approx 4\text{Å}$. It is generally accepted that the important electrons can be described as $Cu$ $d_{x^2-y^2}$ electrons (actually, complicated combinations of $Cu$ and $O$ and perhaps other states; the $d_{x^2-y^2}$ should be understood in the $|\psi_L\rangle$ sense discussed above). Superconductivity emerges upon doping an insulating 'parent compound' in which there is one hole per $CuO2$ unit. Band theory [61] predicts that the low energy electronic degrees of freedom reside in a single band made up mainly of the antibonding combination of $d_{x^2-y^2}$-symmetry $Cu$ and $O$ orbitals. The best tight binding fit to the two dimensional band structure corresponds to a nearest-neighbor hopping of magnitude $t \approx 0.40eV$ and a second neighbor hopping $t' \approx 0.1eV$ so that the band theory dispersion is

$$\varepsilon_p = -2t(\cos(p_x a) + \cos(p_y a)) + 4t' \cos(p_x a) \cos(p_y a)$$

(1.53)

corresponding to a band fermi velocity of about $4eV\text{–Å}$ and a band kinetic energy (equal to the Drude weight) of about $K_{\text{band}} = 0.28eV$.

Consider the antiferromagnetic insulating end materials $La_2CuO_4$ and $Nd_2CuO_4', 'parent compounds' of the hole and electron doped cuprates respectively. Experimental conductivities [62] are shown in Figs (1.11, 1.12). One expects that because the electronic states of the different rare earths lie very far from the chemical potential [61], the optical spectra of these materials should be very similar. In fact, differences are evident in the frequency range $\omega > 3eV$. However, both materials are insulators with gaps of approximately $2eV$. The optical absorption beginning at $\omega = 2eV$ is attributed to optical excitations to the 'upper Hubbard band' or 'charge transfer band'. Evidently a higher energy feature (perhaps at $\omega \sim 5eV$) produces a 'tail' of absorption which extends down to lower energies. It seems reasonable to attribute the $5eV$ absorption to 'non-bonding' oxygen states which are not of fundamental interest for the physics of high $T_c$, however in the $La_2CuO_4$ sample it does not seem possible to separate this absorption from the 'upper hubbard band' absorption of physical interest. In the $Nd_2CuO_4$ material one might ar-
Figure 1.11. Evolution of conductivity of electron doped high temperature superconductors with doping, from [62]

gue that the high-frequency absorption in the most highly doped sample is representative of the 'tail' of the 5eV absorption. Subtracting this from the measured conductivity, using the in-plane lattice constant of 4Å and an out of plane lattice constant of 6Å and integrating the difference yields a spectral weight of 0.15eV, approximately half of the band structure value, suggesting a $U$ of about $9t \approx 3eV$. This value of $U$ gives a gap of about $5t$, comparable to the observed gap. Thus one may conclude that (with some ambiguities) the optical data are consistent with that expected from the Hubbard model in the intermediate correlation regime. However, the data are in an awkward frequency regime and are subject to some uncertainties, and the estimates are obviously very rough.

As one dopes away from the insulator, low frequency spectral weight appears as shown in Figs (1.11, 1.12). Some fraction of this weight comes from the $\omega \sim 2eV$ 'upper Hubbard band' region and some fraction appears to come from much higher energies. A quantitative understanding of the scales over which the spectral weight is redistributed has not yet been achieved.

4.2 CMR materials

The 'colossal' magnetoresistance (CMR) manganites occur in a variety of crystal structures but share the common feature that the mobile
electrons arise from $Mn$ $e_g$ symmetry d-levels and are very strongly coupled to $S = 3/2$ core spins composed of electrons residing in $Mn \, t_{2g}$ levels. They are important in the present context because (as explained above) the large value of the carrier-core-spin coupling means that the 'kinetic energy' is temperature dependent, allowing nontrivial tests of the Peierls approximation and the restricted sum rule.

The prototypical compounds are the pseudocubic manganese perovskites $Re_{1-x}Ak_xMnO_3$ (here $Re$ is a rare earth such as $La$ or $Pr$ and $Ak$ is a divalent alkali such as $Ca$ or $Sr$). The $x = 0$ 'parent compounds' are large gap insulators. The insulating behavior is mainly [16] due to a large-amplitude spatially coherent Jahn-Teller distortion. With doping the distortion is removed (for most choices of $Re$ and $Ak$--a few compounds remain insulating at all dopings) and for $x$ in the range
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0.3 – 0.5 a ferromagnetic metallic ground state results. Unfortunately, there are important and still ill-understood sample (especially surface) preparation issues which dramatically affect optical data obtained in metallic samples [63]. Roughly speaking, the higher the conductivity (at low temperatures) the better the sample and surface preparation. The data discussed here were obtained by Quijada and Simpson in the group of Drew at the University of Maryland using annealed films [64]. The conductivity of these films is comparable to the best conductivities obtained by other groups, but the data should still be regarded as subject to possible correction.

The band structure has been calculated [65, 66]: the conduction bands are derived from two $e_g$ orbitals on each Mn site and are reasonably well described by a nearest neighbor tight binding model [16, 54] (which, for example, reproduces almost exactly the band theory approximation to the specific heat value quoted for $x = 0.3$ manganite in [66]). (Note also that the the value of the Drude plasma frequency $\Omega_p = 1.9\text{eV}$ quoted for the $x = 0.3$ manganite in [66] is in error [67]). It is convenient to adopt a Pauli matrix notation in which the up state is the $[x^2 - y^2]$ orbital and the down state is the $[3z^2 - r^2]$ orbital. Then the basic hopping Hamiltonian is a 2x2 matrix given by

$$\varepsilon = \varepsilon_0(p) + \vec{\varepsilon} \cdot \vec{\tau}$$

with $\vec{\tau}$ the usual Pauli matrices and

$$\varepsilon_0(p) = -t(\cos(px) + \cos(py) + \cos(pz))$$

and $\vec{\varepsilon} = (\varepsilon_x, 0, \varepsilon_z)$ with

$$\varepsilon_x(p) = -\frac{\sqrt{3}t}{2}(\cos(px) - \cos(py))$$

$$\varepsilon_z(p) = t(\cos(pz) - \frac{1}{2}(\cos(px) + \cos(py))) + \Delta_{ef}$$

where $\Delta_{ef}$ is a 'crystal field' energy splitting arising from a tetragonal distortion away from cubic symmetry (as occurs, e.g. in the layered manganites).

The energy eigenvalues are

$$E_{\pm} = \varepsilon_0 \pm \sqrt{\varepsilon_x^2 + \varepsilon_z^2}$$

Note that along the zone diagonals ((1,1,1) and symmetry related) $\varepsilon_{x,z} = 0$ so the two bands are degenerate and along the line to any cube face ((1,0,0) and equivalent) one of the two bands is flat. For this band structure the electron Green function is

$$G(z,p) = (z - \Sigma(p,z) - \varepsilon(p))^{-1}$$
The current operator following from the Peierls approximation is (for currents in the $z$ direction)
\[ j_z = -t \sin(p_z)(1 + \tau_x) \]  
(1.60)

The kinetic energy for motion along one of the cartesian directions is
\[ K = -\frac{1}{3} Tr[\varepsilon_p G] \]  
(1.61)

In the band structure corresponding to the ferromagnetic state of the cubic materials, the chemical potential for $x = 0.3$, $\mu_{0.3} \approx -1$ and the corresponding 'band' kinetic energy, obtained by evaluating Eq (1.61) is
\[ K_{\text{band}}(\mu = -1) = -0.45t = -0.28eV \]  
(1.62)

In the momentum-independent self energy approximation the conductivity is
\[ \sigma(i\Omega) = \frac{1}{i\Omega} T \sum_{i\omega_n} \int \frac{d^3p}{(2\pi)^3} Tr[j_p G(i\omega + i\Omega, p) j_p G(i\omega, p)] \]  
(1.63)

This conductivity includes both 'Drude' and interband terms. The kinetic energy corresponding to the Drude part of the conductivity is (for the widely studied $1/3$ doping level)
\[ K_{\text{Drude, x=0.3}} = 0.32t = 0.2eV \]  
(1.64)

Ahn and the author [16] evaluated Eq (1.63) for the insulating 'parent compound' $LaMnO_3$, in which a large-amplitude spatially coherent Jahn-Teller distortion occurs which is sufficient to explain the insulating behavior [65] (although band theory somewhat underpredicts the gap [16]) With doping the distortion is removed and a ferromagnetic metallic ground state results. Their calculation used the nearest neighbor tight binding parametrization of the band theory, along with realistic values for the level splitting caused by the Jahn-Teller distortion, as well as a 'Hubbard U' (treated in the Hartree approximation) and various estimates for the Kondo coupling $J$. The calculations, while in reasonable agreement with data taken at room temperature, predict a larger than observed increase in spectral weight as the temperature is lowered. The band theory conductivity of $LaMnO_3$ was also calculated by Soloviev et al using LMTO methods to evaluate actual wave function overlaps. Remarkably, the results (while quite close to the tight binding results for the magnitude of the insulating gap) indicate spectral weights about
Figure 1.13. Measured conductivity of 'CMR' manganites, from [64]
four times smaller than the spectral weights predicted by the Peierls approximation. The source of this discrepancy has not been determined; resolving it is an important issue for future research.

The theoretical conductivity of the metallic materials has not been investigated in such detail. Shiba and collaborators studied the \( T = 0 \) conductivity of the fully polarized ferromagnetic state [68]. Chattopadhyay and the author [54] considered changes in spectral weight between \( T_c \) and \( T = 0 \) and related these to the magnetic transition temperatures.

Fig (1.13) shows the optical conductivity for several different \( x = 0.3 \) doped (ferromagnetic metallic ground state) manganites obtained by [64]. A strong dependence of both the form and the integrated area of the low frequency conductivity is evident. Fig. (1.14) shows the integrated area, for different temperatures. A change of spectral weight with temperature is evident; presumably this is related to double exchange.

To analyse the data in a satisfactory manner one must identify the contribution to the observed conductivity coming from the conduction bands. Use of the magnitude and temperature dependence of the spectral weight provides enough information to do this [54, 64]. The temperature dependence must arise from the double-exchange physics and thus from the conduction band. The total conduction band spectral weight must be less than the band kinetic energy, the value of the \( T = 0 \) spectral weight and the change between low \( T \) and \( T > T_c \) must be large enough to explain the observed \( T_c \) and the change in spectral weight cannot be more than about half of the low-\( T \) weight. These considerations led to the conclusion [64] that in the most metallic material, \( \text{La}_{0.7}\text{Sr}_{0.3}\text{MnO}_3 \) at low \( T \) the conduction band contribution essentially exhausts the bound imposed by the band theory, so the 'Hubbard – \( U \)' effects are evidently weak!

Indications are [15, 64] that an additional interaction (probably the electron-phonon interaction, which causes charge ordering in some materials) leads to additional structure in the conductivity (but not to significant changes in the low frequency spectral weight). Presently available data suggest that these extra interactions are poorly described by a momentum-independent self energy. In particular, if the self energy is momentum independent, then the mass enhancement inferred from the specific heat should be the same as the mass enhancement inferred from the renormalization of the Drude weight. However, recent experimental results suggest that the Drude weight is more strongly reduced (relative to band theory) than is the fermi velocity [69].
Figure 1.14. Measured spectral weight of 'CMR' manganites, from [64]
5. Conclusions

I hope in this brief survey to have conveyed some of the basic ideas in
the theory of optical conductivity of correlated electron systems, along
with some of the open theoretical challenges and to have shown how
these ideas are used in practice. Most importantly, I hope to have given
the reader at least a glimpse of the power of the technique for elucidating
correlated electron physics.

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