PHY W3003: Final Examination

May 8, 2015

Please answer all questions, and **show your work** as partial credit will be given.

You may bring to the exam one sheet (both sides) of paper. Use of calculators or other electronic devices is **not allowed** during the examination. Where numerical answers are requested, answers to one significant figure are acceptable.

1 (20 points) An *undamped* oscillator of mass m and natural frequency ω_0 is initially at rest at its equilibrium position $x = 0$. For times $t \geq 0$ it is subject to a force $F(t) = f_0 e^{-kt}$ with f_0 and k positive real constants.

(a) (5 points) Please write the equation of motion of the oscillator for times $t > 0$.

$$\ddot{x} + \omega_0^2 x = \frac{f_0}{m} e^{-kt} \Theta(t)$$

(b) (5 points) Please give the most general solution of the equation of motion for times $t > 0$.

$$x(t) = Acos\left(\omega_0 t + \phi\right) + \frac{\frac{f_0}{m} e^{-kt}}{k^2 + \omega_0^2} \Theta(t)$$

(c) (5 points) Using the initial conditions please find the motion $x(t)$ for all $t > 0$ in terms of ω_0, f_0 and k.

We require

$$\begin{aligned} 0 = x(t=0) &= Acos\phi + \frac{\frac{f_0}{m}}{k^2 + \omega_0^2} \\ 0 = \dot{x}(t=0) &= -A\omega_0 sin\phi - \frac{\frac{f_0 k}{m}}{k^2 + \omega_0^2} \end{aligned}$$

Thus

$$\begin{aligned} Acos\phi &= -\frac{\frac{f_0}{m}}{k^2 + \omega_0^2} \\ Asin\phi &= -\frac{\frac{f_0}{m}\frac{k}{\omega_0}}{k^2 + \omega_0^2} \end{aligned}$$

so

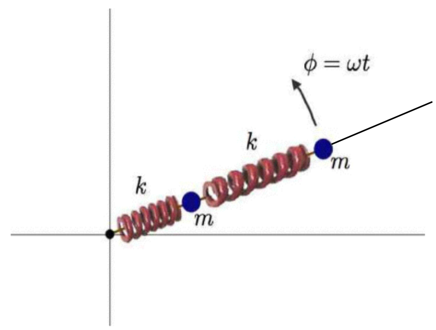
$$\begin{aligned} A &= \frac{\frac{f_0}{m}}{\omega_0 \sqrt{k^2 + \omega_0^2}} \\ tan\phi &= \frac{k}{\omega_0} \end{aligned}$$

(d) (5 points) As time $t \to \infty$ the energy of the oscillator approaches a constant value E_∞. Please find E_∞.

As $t \to \infty$ the term proportional to the driving force drops out. The energy is then just the energy of the free oscillation which is

$$E = \frac{1}{2} m v_{max}^2 = \frac{m A^2 \omega_0^2}{2} = \frac{\frac{f_0^2}{m}}{2\left(k^2 + \omega_0^2\right)}$$

2 (20 points) Two masses and two springs are constrained to move along a massless rod which is externally driven to rotate with angular velocity ω about a fixed point on a horizontal surface, as shown in the Figure. To be specific, one mass is connected to the origin by a spring of force constant k and to the other mass by a spring also of force constant k. The other mass is connected to the first mass, by a spring of force constant k. The equilibrium length of each spring is a, the force constant of each spring is k and the mass of each particle is m.



(a) (10 points) Choose as generalized coordinates the radial distances $r_{1,2}$ of particles 1 and 2 from the origin. Find the Lagrangian $\mathcal{L}(r_1, \dot{r}_1, r_2, \dot{r}_2, t)$.

The kinetic energy is

$$K = \frac{m}{2}\left(\dot{r}_1^2 + r_1^2\omega^2 + \dot{r}_2^2 + r_2^2\omega^2\right)$$

while the potential energy is

$$V = \frac{k}{2}(r_1 - a)^2 + \frac{k}{2}(r_2 - r_1 - a)^2$$

so

$$\mathcal{L} = K - V$$

(b) (10 points) Find the equilibrium extensions of the springs, as a function of ω. *At equilibrium $\ddot{r}_1 = \ddot{r}_2 = 0$ so we must solve*

$$\begin{aligned} m\ddot{r}_1 &= m\omega^2 r_1 - k(r_1 - a) + k(r_2 - r_1 - a) = 0 \\ m\ddot{r}_2 &= m\omega^2 r_2 - k(r_2 - r_1 - a) = 0 \end{aligned}$$

or

$$\begin{aligned} 0 &= \left(2k - m\omega^2\right) r_1 - k r_2 \\ ka &= \left(k - m\omega^2\right) r_2 - k r_1 \end{aligned}$$

Thus

$$\begin{aligned} r_1 &= \frac{k}{2k - m\omega^2} r_2 \\ ka &= \left(k - m\omega^2 - \frac{k^2}{2k - m\omega^2}\right) r_2 \\ &= \frac{k^2 - 3km\omega^2 + m^2\omega^4}{2k - m\omega^2} r_2 \end{aligned}$$

so

$$r_1 = \frac{k^2}{k^2 - 3km\omega^2 + m^2\omega^4}a \qquad r_2 = \frac{2k^2 - km\omega^2}{k^2 - 3km\omega^2 + m^2\omega^4}a;$$

3 A particle of mass m moves in two dimensions subject to the potential

$$U(r) = U_0 ln \frac{r^2}{r^2 + b^2}$$

which depends only on the radial distance of the particle from an origin.

(a) (5 points) Sketch the potential energy

Suppose that the particle has angular momentum L_z measured about an axis passing through the origin and perpendicular to the plane of motion.

(b) (10 points) What is the largest L_z for which a stable circular orbit is possible? *The effective potential is*

$$U_{eff}(r) = \frac{L_z^2}{2mr^2} + U_0 ln \frac{r^2}{r^2 + b^2}$$

Circular motion is possible if for some $r = r^\star$ $dU/dr = 0$ *and is stable if at this point* $d^2U/dr^2 > 0$. *We have*

$$\begin{aligned} \frac{dU}{dr} &= -\frac{L_z^2}{mr^3} + 2U_0\left(\frac{1}{r} - \frac{r}{r^2+b^2}\right) = -\frac{L_z^2}{mr^3} + \frac{2U_0 b^2}{r(r^2+b^2)} \\ &= \frac{1}{r^3(r^2+b^2)}\left(-\frac{L_z^2}{m}(r^2+b^2) + 2U_0 b^2 r^2\right) = \frac{\left(2U_0 b^2 - \frac{L_z^2}{m}\right) r^2 - \frac{L_z^2}{m} b^2}{r^3(r^2+b^2)} \end{aligned}$$

So we see that if $U_0 > \frac{L_z^2}{2mb^2}$ *then circular motion is possible (solution at positive* r^2*):*

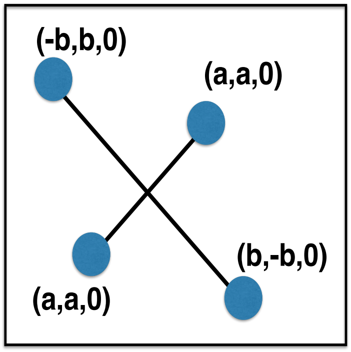
$$(r^\star)^2 = \frac{\frac{L_z^2}{2mb^2}}{U_0 - \frac{L_z^2}{2mb^2}}$$

We can evaluate the second derivative and substitute $r = r^\star$, *but it is easier to note that there is only one extremum, and that this must be a minimum because if* $U_0 > \frac{L_z^2}{2mb^2}$ *then the potential goes down as* r *moves in from* ∞ *but goes up as* $r \to 0$ *so what is in between must be a minimum.*

(c) (5 points) For L_z less than the value you found in (b) sketch the phase space orbits in the $r, \dot{r}$ plane, indicating the bound, unbound and separatrix orbits (if any), and give the energy of the separatrices (if any).

The energy of the separatrix is 0

4 Consider four points of equal mass m lying in the x-y plane at positions $(a,a,0)$, $(-b,b,0)$, $(-a,-a,0)$ and $(b,-b,0)$ and connected to the origin by rigid massless rods fixed at right angles to each other.



(a) (5 points) If $b = 2a$ please find the 3×3 inertia tensor describing rotations about axes passing through the origin, using the (x,y,z) coordinate system given in the figure and expressing your answer in terms of m and a.

(b) (5 points) Please find the principle axes and moments of inertia.

The inertia tensor for general a, b is

$$\mathbf{I} = \begin{pmatrix} 2m(a^2+b^2) & 2m(b^2-a^2) & 0 \\ 2m(b^2-a^2) & 2m(a^2+b^2) & 0 \\ 0 & 0 & 4m(a^2+b^2) \end{pmatrix}$$

The principal axes are

$$\begin{aligned} \hat{v}_1 &= \hat{e}_z; \qquad Eigenvalue\ \lambda_1 = 4m(a^2+b^2) \\ \hat{v}_2 &= \frac{\hat{e}_x+\hat{e}_y}{\sqrt{2}}; \qquad Eigenvalue\ \lambda_2 = 4mb^2 \\ \hat{v}_3 &= \frac{\hat{e}_x-\hat{e}_y}{\sqrt{2}}; \qquad Eigenvalue\ \lambda_3 = 4ma^2 \end{aligned}$$

(c) (5 points) Now suppose the object is rotating about an axis passing through the origin, with angular momentum $\vec{L}$ whose components (in the coordinate system above) are $L_x = L$, $L_y = L$, $L_z = 0$. Please find the the 3×3 inertia tensor describing rotations about axes passing through the origin, using the (x,y,z) coordinate system given in (a) and expressing your answer in terms of m and a. *Note that I am asking you to find the components of the inertia tensor in the space frame.*

The angular momentum is directed along the "2" principal axis, so the angular velocity is

$$\omega = \frac{L}{4mb^2} = \frac{L}{16ma^2}$$

The rotation means that the points at distance b are time dependent, with coordinates

$$\vec{r} = b\left(cos\omega t, cos\omega t, \sqrt{2}sin\omega t\right) = 2a\left(cos\omega t, cos\omega t, \sqrt{2}sin\omega t\right)$$

The inertia tensor is then

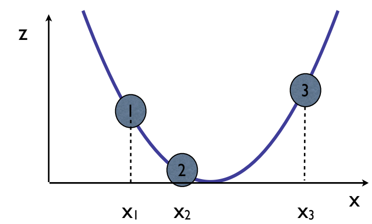
$$\mathbf{I} = \begin{pmatrix} 2m(a^2+b^2(cos^2\omega t+2sin^2\omega t)) & 2m(b^2cos^2\omega t-a^2) & -\sqrt{2}b^2sin\omega t cos\omega t \\ 2m(b^2cos^2\omega t-a^2) & 2m(a^2+b^2(cos^2\omega t+2sin^2\omega t)) & \sqrt{2}b^2sin\omega t cos\omega t \\ -\sqrt{2}b^2sin\omega t cos\omega t & \sqrt{2}b^2sin\omega t cos\omega t & 4m(a^2+b^2cos^2\omega t) \end{pmatrix}$$

(d) (5 points) While rotating as in (c), b is instantaneously decreased from $b = 2a$ to $b = a/2$. Please find the angular velocity vector immediately after the change. Please also state whether the subsequent motion is stable (meaning that if the angular momentum is slightly changed, the angular velocity will remain close to the value you just found, or whether the object will start to tumble randomly).

$$\omega \to \frac{L}{4mb^2} = \frac{L}{ma}$$

The motion is unstable because the rotation is now around the axis with intermediate moment of inertia.

5 Three particles of mass m slide without friction on a parabolic track $z = x^2/2b$ with b a length and are subject to the force of gravity $-mg\hat{z}$. Adjacent particles are coupled by springs of spring constant k and equilibrium length a. *Assume for simplicity that the spring energy depends only on the differences between x coordinates, e.g.* $V_{12} = \frac{k}{2}\left(x_2 - x_1 - a\right)^2$.



(a) (5 points) Please write the Lagrangian of the system, using as generalized coordinates the x positions of the three particles.

The kinetic energy of a particle is $\frac{m}{2}\left(\dot{z}^2 + \dot{x}^2\right)$ *and the constraint means* $z = x^2/2$ *so* $\dot{z} = x\dot{x}$ *so*

$$K = \frac{m}{2}\left((1 + x_1^2)\dot{x_1}^2 + (1 + x_2^2)\dot{x_2}^2 + (1 + x_3^2)\dot{x_3}^2\right)$$

The potential energy is the sum of the spring and gravitational energies

$$V = \frac{k}{2}\left((x_2 - x_1 - a)^2 + (x_3 - x_2 - a)^2\right) + \frac{mg}{2}\left(x_1^2 + x_2^2 + x_3^2\right)$$

and

$$\mathcal{L} = K - V$$

(b) (5 points) Please write the equations that determine the configuration of minimum energy and solve them (*hint: on physical grounds we expect this configuration is symmetric about* $x = 0$)

The equations are $\partial V/\partial x_1 = \partial V/\partial x_2 = \partial V/\partial x_3 = 0$, *i.e.*

$$\begin{aligned} -k(x_2 - x_1 - a) + mgx_1 &= 0 \\ k(x_2 - x_1 - a) - k(x_3 - x_2 - a) + mgx_2 &= 0 \\ k(x_3 - x_2 - a) + mgx_3 &= 0 \end{aligned}$$

On physical grounds we expect that at the minimum $x_2 = 0$ *and* $x_1 = -x_3$; *substituting, we have*

$$\begin{aligned} -k(x_3 - a) - mgx_3 &= 0 \\ k(x_3 - a) - k(x_3 - a) &= 0 \\ k(x_3 - a) + mgx_3 &= 0 \end{aligned}$$

so we see that this guess provides a solution, which is

$$x_3 = a^\star = \frac{k}{k + mg}$$

(c) (5 points) Please find the Lagrangian that describes small oscillations about the configuration that minimizes the energy.

Writing

$$\begin{aligned} x_1 &= -a^\star + \delta_1 \\ x_2 &= \delta_2 \\ x_3 &= a^\star + \delta_3 \end{aligned}$$

and working to second order in δ we get

$$\begin{aligned} K &= \frac{m}{2}\left((1+(a^\star)^2)\dot{\delta_1}^2 + \dot{\delta_2}^2 + (1+(a^\star)^2)\dot{\delta_3}^2\right) \\ V &= \frac{mg+k}{2}\left(\delta_1^2+\delta_3^2\right) + \frac{mg+2k}{2}\delta_2^2 - k\left(\delta_1\delta_2 + \delta_3\delta_2\right) \end{aligned}$$

(d) (5 points) Is the "uniform motion" configuration in which all three particles move back and forth together without changing their relative positions a normal mode? Give a yes or no answer and a mathematical argument which justifies your answer.

No. If I set $\delta_1 = \delta_2 = \delta_3 = \delta$ then I find

$$V(\delta) = \frac{3mg}{2}\delta^2 \neq 0$$

In other words the system is not translationally invariant.