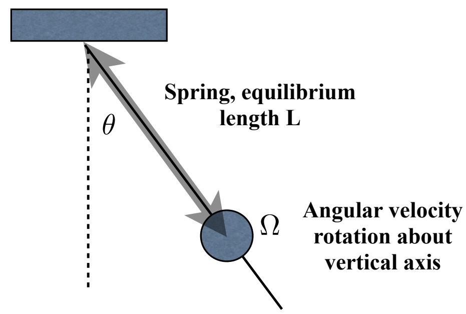
Solutions to PHY W3003: Practice Midterm

1 (20 points) A particle of mass m slides without friction along a rigid rod which rotates about a vertical axis with fixed polar angle θ with *time-varying* azimuthal angular velocity $\Omega(t) = A\, cos\Omega_1\, t$. The particle is connected to the top of the rod by a spring of force constant k and equilibrium length L. (See Figure).



(a) (5 points) Please write down the Lagrangian of this system using as dynamical variable the length l of the spring.

The particle's x,y,z position and velocity are

$$\begin{aligned} x &= lsin\theta cos\Omega(t) & \dot{x} = \dot{l}sin\theta cos\Omega(t) - lsin\theta sin\Omega\dot{\Omega} \\ y &= lsin\theta sin\Omega(t) & \dot{y} = \dot{l}sin\theta sin\Omega(t) + lsin\theta cos\Omega\dot{\Omega} \\ z &= -lcos\theta & \dot{z} = -\dot{l}cos\theta \end{aligned}$$

so the Lagrangian is

$$\mathcal{L} = \frac{m}{2}\left(\dot{l}^2 + l^2 sin^2\theta\dot{\Omega}^2\right) + mglcos\theta - \frac{k}{2}\left(l - L\right)^2$$

(b) (5 points) Please find the equation of motion

$$m\ddot{l} = mgcos\theta + mlsin^2\theta\dot{\Omega}^2 - k\left(l - L\right)$$

(c) (10 points) For some ranges of Ω_1 and A the motion may become unbounded (extension of the spring can become arbitrarily large). Please find these ranges

Using the explicit form of $\Omega(t)$ and the formula for $sin^2\Omega$ we obtain

$$m\ddot{l} = mgcos\theta + \frac{mlsin^2\theta A^2\Omega_1^2}{2}\left(1 - cos(2\Omega_1 t)\right) - k\left(l - L\right)$$

Rearranging gives

$$m\ddot{l} = -\left(k - \frac{msin^2\theta A^2\Omega_1^2}{2}\right)l + mgcos\theta + kL - \left(\frac{msin^2\theta A^2\Omega_1^2}{2}cos(2\Omega_1)\right)l$$

Thus leaving aside the time dependent terms we have a harmonic oscillator with frequency

$$\omega^2 = \omega_0^2 - \frac{sin^2\theta A^2\Omega_1^2}{2}$$

and $\omega_0^2 = k/m$. *If* ω^2 *goes negative, i.e. if* $\frac{sin^2\theta A^2\Omega_1^2}{2} > \omega_0^2$, *then the motion becomes unbounded.*

Also if $2\Omega_1 = \omega$ **then we have an oscillator driven at its resonance frequency and motion becomes unbounded.**

2 (20 points) A particle of mass m moves in three dimensions subject to the potential $U(x, y, z) = \frac{k}{2}\left(x^2 + y^2\right)$.

(a) (5 points) Please state whether energy and the different components of momentum and angular momentum are conserved and *give your reasons.*

Energy is conserved because the system is subject to a time-independent potential. The z *component of angular momentum is conserved because the system is rotationally invariant about the* z *axis. The* z *component of momentum is conserved because the system is translationally invariant along* z.

(b) (10 points) Suppose that the particle is subject to a frictional force $\vec{F} = -\gamma\vec{v}$, and that at time $t = 0$ the particle is at position $x = x_0$, $y = 0$ and $z = 0$ with velocity $\vec{v} = v_z\hat{z}$. Please find the motion at all subsequent times *The equations of motion are*

$$
\begin{aligned}
m\ddot{x} + \gamma\dot{x} + kx &= 0 \\
m\ddot{y} + \gamma\dot{y} + ky &= 0 \\
m\ddot{z} + \gamma\dot{z} &= 0
\end{aligned}
$$

The solutions, consistent with boundary conditions, are (for the underdamped case

$$
\begin{aligned}
x(t) &= C_+e^{\omega_+ t} + C_-e^{\omega_- t} \\
C_+ &= -\frac{x_0\omega_-}{\omega_+ - \omega_-} \qquad C_- = \frac{x_0\omega_+}{\omega_+ - \omega_-} \\
\omega_+ &= \frac{-\gamma + \sqrt{\gamma^2 - 4mk}}{2m} \qquad \omega_- = \frac{-\gamma - \sqrt{\gamma^2 - 4mk}}{2m} \\
y(t) &= 0 \\
z(t) &= \frac{v_z m}{\gamma}\left(1 - e^{-\frac{t\gamma}{m}}\right)
\end{aligned}
$$

(c) (5 points) Determine how much work the particle does on the degrees of freedom providing the friction over the entire time period $t = 0$ to $t = \infty$. (*Note that this answer can be obtained even if you dont have the full solution requested in (b).*

Initial energy is $mv^2/2 + kx_0^2/2$. *Final energy is* 0. *Thus total work done is equal to the initial energy.*

3 (20 points) A particle of mass m moves in three dimensions subject to the potential

$$U(r) = \frac{U_8}{8r^8} - \frac{U_5}{5r^5}$$

with U_8 and U_5 both positive.

(a) (10 points) Please determine the range of angular momenta for which circular motion is possible and the radius and period of the resulting orbit.

The effective potential is

$$U_{eff}(r) = \frac{L^2}{2mr^2} + \frac{U_8}{8r^8} - \frac{U_5}{5r^5}$$

Circular motion is possible if $dU_{eff}/dr = 0$, i.e. if

$$0 = \frac{L^2}{mr^3} + \frac{U_8}{r^9} - \frac{U_5}{r^6}$$

i.e. if

$$0 = \frac{L^2}{m}r^6 - U_5 r^3 + U_8; \quad \rightarrow r^3_{circ} = \frac{U_5 \pm \sqrt{U_5^2 - 4\frac{U_8 L^2}{m}}}{\frac{L^2}{m}}$$

Thus a solution exists if

$$\frac{L^2}{m} > \frac{U_5^2}{4U_8}$$

The solution is stable if

$$0 < \frac{d^2U_{eff}}{dr^2} = \frac{3L^2}{mr^4} + \frac{9U_8}{r^{10}} - \frac{6U_5}{r^7} = \frac{1}{r^{10}}\left(\frac{3L^2}{m}r^6 - 6U_5 r^3 + 9U_8\right)$$

Because the effective potential increases as r is decreased from infinity the larger r extremum is a maximum so the radius of stable circular motion is

$$r^3_{circ,stable} = \frac{U_5 - \sqrt{U_5^2 - 4\frac{U_8 L^2}{m}}}{2\frac{L^2}{m}}$$

and the period is

$$T = \frac{2\pi r^2_{circ,stable}}{L}$$

(b) (5 points) Suppose the motion of a particle moving in the circular orbit found in (a) is slightly perturbed, so that at time $t = 0$ the value of the radial coordinate is changed to $r \rightarrow r_{circ} + \delta$ with δ small, but no velocities are changed. Please determine, in terms of δ, U_8, U_5 r_{circ} and m the maximum value of the radial velocity $\dot{r}$. *for this problem you may work to lowest order in δ) and you dont need to simplify the expressions.*

In the radial direction the perturbed motion is harmonic, with mass m and restoring force $k = d^2U_{eff}/dr^2$ implying frequency $\omega_{rad} = \sqrt{\frac{d^2U_{eff}/dr^2}{m}}$ so the solution to the radial equation is

$$r(t) = r_{circ} + \delta cos\left(\omega_{rad}t\right)$$

implying that the velocity is

$$v_r(t) = -\delta\omega_{rad} sin\left(\omega_{rad}t\right)$$

so that the maximum velocity is

$$v_{max} = \delta\omega_{rad}$$

(c) (5 points) Is the resulting orbit closed? (Give reasons).

The resulting orbit will be closed if the oscillation frequency in the radial direction is an integer multiple of the orbital period. For small δ the condition is

$$\omega_{rad} = \frac{2\pi n}{T}$$

with T the period found in part (a)