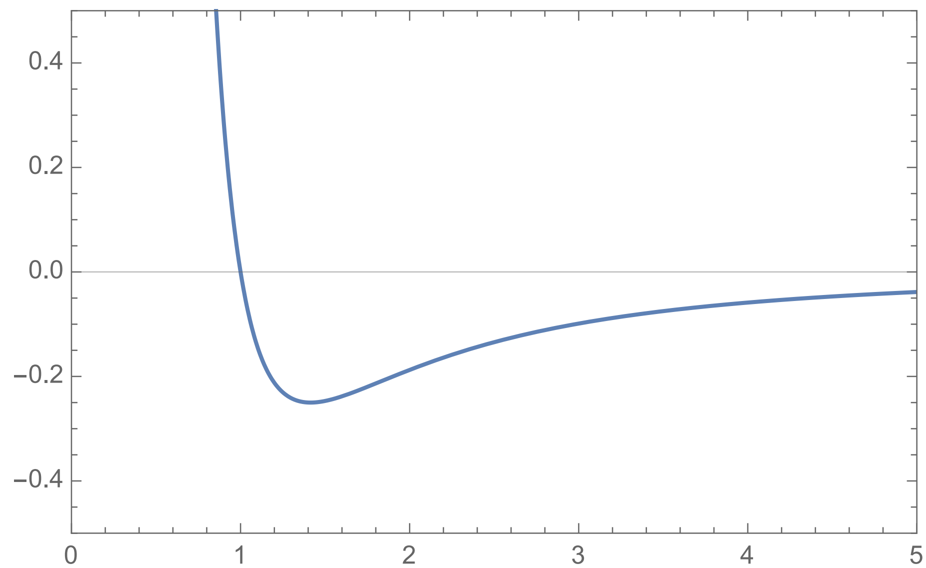


FIGURE 1. Potential for $U_0 = U_1 = 1$

PHY W3003: Midterm Examination I Feb 18, 2015

Please answer all questions, and **show your work** as partial credit will be given.

You may bring to the exam one sheet (both sides) of paper. Use of calculators or other electronic devices is **not allowed** during the examination. Where numerical answers are requested, answers to one significant figure are acceptable.

1 (20 points): A particle moves in the half plane $x > 0$ subject to the one-dimensional potential

$$U(x) = \frac{U_0}{x^4} - \frac{U_1}{x^2}$$

wtih U_0 abd U_1 positive.

(a) (5 points) Sketch $U(x)$. Identify the locations of any equilibrium points and classify them as stable or unstable. Be sure that your sketch shows the proper behavior at $x \to \infty$

The minimum is found by setting $dU/dx = 0$, so

$$x = \sqrt{\frac{2U_0}{U_1}}$$

This is a stable equilibrium (second derivative is positive). At $x \to \infty$ $U \to 0$

(b) (10 points) Sketch the phase space orbits, indicating the bound, unbound and separatrix orbits, and give the energy of the separatrix orbits.

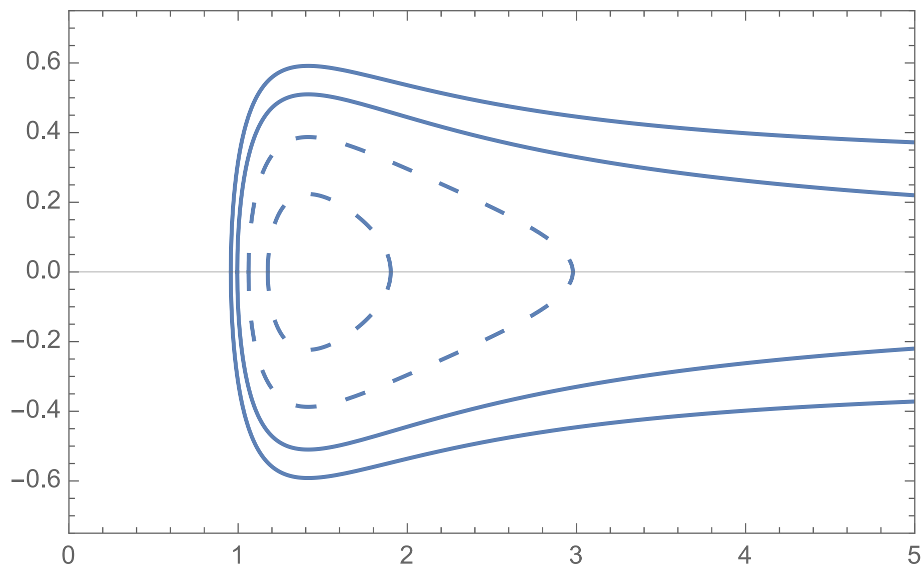


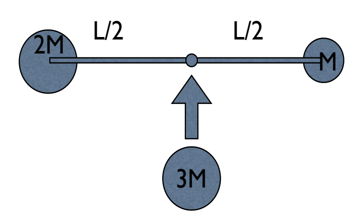
FIGURE 2. Phase portraits for potential sketched in fig 1. Bound orbits are shown as dashed lines and unbound as solid lines

the energy of separatrix orbit is 0.

(c) (5 points). Suppose $U_1 = 2U_0$. Please sketch all orbits of energy $-U_0$.

At $U_1 = 2U_0$ the potential minimum is at $x = 1$ and the value of the potential at the minimum is $U = -U_0$ so the orbit is a point, at the potential minimum

2 (20 points) A dumbbell consists of two particles, one of mass M and one of mass $2M$, connected by a rigid massless rod of length L. At time $t = 0$ the dumbbell suffers an instantaneous and fully *elastic* collision with a particle of mass $3M$ which strikes the center of the dumbbell and is moving perpendicular to it (see figure).



I will first analyse the collision in detail, and then answer the questions. Note that most of the answers required are qualitative and dont depend on the the full solution.

We use conservation of energy, angular momentum and linear momentum to determine the velocities v_R of the right-hand ball of the dumbbell, v_L of the left hand ball of the dumbbell, and v_F the final velocity of the incident particle, in terms of the initial velocity v of the incident particle.

The center of mass of the dumbell at a distance $-L/6$ relative to the point of impact.

I choose to define angular momentum with respect to an axis perpendicular to the page and passing through the point of collision. Before the collision

$$P_I = 3Mv \qquad L_I = 0 \qquad E_I = \frac{3Mv^2}{2}$$

Immediately after the collision all velocities are directed parallel to the initial velocity. The total linear momentum after the collision is

$$2MV_L + MV_R + 3MV_F = 3Mv$$

The total angular momentum after the collision is the sum of the angular momentum associated with rotations around the center of mass and the angular momentum obtained by treating the dumbell as a particle with position equal to that of the COM. Thus

$$\frac{2L}{3}MV_R - \frac{L}{3}(2M)V_L - (\frac{L}{6})\left(2MV_L + MV_R\right) = 0$$

or

$$V_R = 2V_L$$

Combining this with the equation for center of mass momentum gives

$$4V_L + V_F = v$$

Finally conservation of energy gives

$$\frac{2MV_L^2}{2} + \frac{MV_R^2}{2} + \frac{3MV_F^2}{2} = \frac{3Mv^2}{2} = \frac{3M\left(V_F + \frac{V_R}{3} + \frac{2V_L}{3}\right)^2}{2}$$

Using $V_R = 2V_L$ and multiplying by $2/M$ gives

$$6V_L^2 + 3V_F^2 = 3\left(V_F + \frac{4V_L}{3}\right)^2$$

implying

$$V_L = 12V_F$$

so that

$$V_F = \frac{v}{17} \qquad L = \frac{8MLv}{17} \qquad P_{dumbbell,COM} = \frac{3Mv}{17}$$

(a) (10 points) After the collision, is the dumbell rotating? If so, in what sense (which direction) and with what angular momentum

The dumbell is rotating anticlockwise, because the collision point is to the right of the center of mass (which is $L/6$ to the left of the point of impact). L is given above.

(b) (10 points) After the collision is the incident particle moving, and if so in what direction? (Note you are not being asked to find the velocity; only to say whether it is zero or nonzero and, if nonzero, what is the direction). *The incident particle is moving forward, because some of the kinetic energy of the incident particle went into making the dumbbell rotate. It's COM velocity and thus momentum is therefore less than it would have been in a no-rotation elastic collision, but we know in the latter case the final velocity of the particle is zero.*

4 (20 points) A particle of mass $M = 1kG$ moving in one dimension is subject to a frictional force proportional to the square root of its velocity, $F_f = -\gamma v^{\frac{1}{2}}$ with $\gamma = 2N - sec^{\frac{1}{2}}/cm^{\frac{1}{2}}$.

(a) (10 points) If the particle is also subject to an additional force $F = 1N$ what is the terminal velocity

The equation of motion is

$$\ddot{x} + 2\left(\dot{x}\right)^{\frac{1}{2}} = 1$$

The terminal velocity is found by balancing the dissipation against the force, so

$$v_T = \frac{1}{4} cm/sec$$

(b) (10 points) For this part set the additional force to zero (but the frictional force of course remains present). Assume that at time $t = 0$ the particle is at position $x = 0$ with velocity $v_0 = 1m/sec$. How far from $x = 0$ is the particle when it stops?

The equation of motion in the absence of the force is

$$M\dot{v} = -\gamma v^{\frac{1}{2}}$$

or

$$\frac{dv}{\sqrt{v}} = -\frac{\gamma}{M} dt$$

Integrating

$$2\sqrt{v(t)} - 2\sqrt{v(t=0)} = -\frac{\gamma t}{M}$$

Thus the time when $v(t) = 0$ is

$$t_{zero} = 2\frac{M}{\gamma} = 1 \ sec$$

the distance travelled is

$$x(t) - x(0) = \int_0^t dt v(t) = \int_0^t dt' \left(\sqrt{v(t=0)} - \frac{\gamma t'}{2M}\right)^2 = v(t=0)t - \sqrt{v(t=0)}\frac{\gamma t^2}{2M} + \frac{\gamma^2}{12M^2} t^3$$

Putting in numbers

$$x_{stop} = \frac{1}{3} cm$$