PHY W3003 2016: Solutions to Midterm Examination II 6

1 (25 points): A particle of mass m is constrained to move on a parabolic track $z = kx^2$ with z the height and is subject to the force of gravity.

(a) (5 points) Please write the Lagrangian of the system, using as generalized coordinate the horizontal position x.

We have

$$\mathcal{L} = \frac{m}{2}\left(\dot{x}^2 + \dot{z}^2\right) - mgz$$

and

$$z = kx^2$$

so

$$\dot{z} = 2kx\dot{x}$$

so

$$\mathcal{L} = \frac{m}{2}\left(\dot{x}^2 + 4k^2x^2\dot{x}^2\right) - mgkx^2$$

(b) (5 points) Please write the Euler-Lagrange equation which determines the time dependence of x

$$m\frac{d}{dt}\left(\dot{x} + 4k^2x^2\dot{x}\right) = 4mk^2\dot{x}^2x - 2mgkx$$

or

$$\left(1 + 4kx^2\right)\ddot{x} + 8k^2x\dot{x}^2 = 4k^2\dot{x}^2x - 2gkx$$

or

$$\left(1 + 4kx^2\right)\ddot{x} + 4k^2x\dot{x}^2 = -2gkx$$

(c) (5 points) If the displacements x are very close to zero the motion is approximately harmonic. Please find the oscillation frequency in terms of k, m and the gravitational constant g *Removing all terms involving higher powers of x than unity we get*

$$\ddot{x} = -2gkx$$

so the oscillator frequency is $\sqrt{2gk}$

(d) (10 points) Suppose now the track is made to rotate with angular frequency ω about the z axis. Please find the critical rotation frequency above which the equilibrium point at $x = 0$ becomes unstable and for rotation frequencies greater than this value find the new equilibrium point (if it exists).

We now have

$$\mathcal{L} = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - mgz$$

writing $x = rcos\omega t$ *and* $y = rsin\omega t$ *we have*

$$\begin{aligned}\dot{x} &= \dot{r}cos\omega t - \omega rsin\omega t \\ \dot{y} &= \dot{r}sin\omega t + \omega rcos\omega t\end{aligned}$$

so using also the constraint

$$z = k(x^2 + y^2) = kr^2 \qquad \dot{z} = 2kr\dot{r}$$

we get

$$\mathcal{L} = \frac{m}{2}\left(\dot{r}^2 + 4k^2r^2\dot{r}^2 + \omega^2 r^2\right) - mgkr^2$$

so

$$m\frac{d}{dt}\left(\dot{r} + 4k^2r^2\dot{r}\right) = 4mk^2\dot{x}^2x + m\omega^2 r - 2mgkr$$

Linearizing as before

$$\left(\ddot{r} + 4k^2r^2\dot{r}\right) = \left(\omega^2 - 2gk\right) r$$

so the critical rotation speed that destabilizes the motion is $\omega = \sqrt{2gk}$.

The effective potential is $\frac{m}{2}\left(\omega^2 - 2gk\right) r^2$; it has no stable equilibrium except 0

2 (25 points) Two particles of masses m_1 and m_2 move subject to the central potential $U = k|\vec{r}_1 - \vec{r}_2|^{\frac{1}{2}}$ with $k > 0$.

(a) (5 points) If the magnitude of the angular momentum measured about an axis passing through the center of mass and perpendicular to the relative velocity of the particles is L, write the equation of governing the time dependence of the relative coordinate $r = r_1 - r_2$.

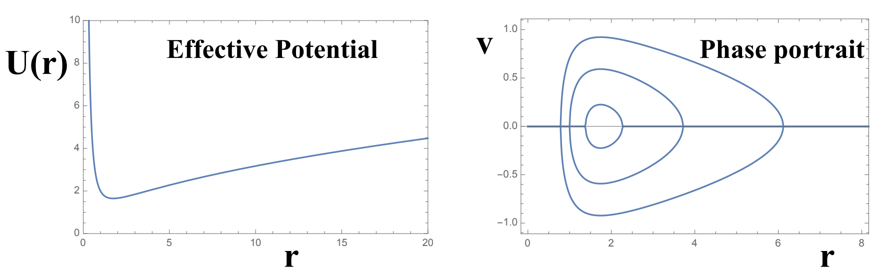
The effective potential governing the relative motion of the particles is

$$U_{eff}(r) = \frac{L^2}{2\mu r^2} + k\sqrt{r}$$

with $\mu = m_1m_2/(m_1 + m_2)$ *so the equation of motion is*

$$\mu\ddot{r} = \frac{L^2}{\mu r^3} - \frac{k}{2\sqrt{r}}$$

(b) (5) sketch the effective potential governing the radial motion of the relative coordinate and also the phase space orbits, identifying bound and unbound orbits and giving the energy of the separatrix.



All orbits are bound because the potential diverges at infinity. There is no separatrix

(c) (5 points) Find the radii at which circular orbits can exist and determine whether the orbits are stable.

The radii at which stable circular orbits can exist are those where the force vanishes or the effective potential has a minimum. From the sketch in (b) we see that there is only one minimum (this can be verified by computing the second derivative), which is given by

$$\frac{L^2}{\mu r^3} = \frac{k}{2\sqrt{r}}$$

i.e.

$$r = \left(\frac{2L^2}{\mu k}\right)^{\frac{2}{5}}$$

(d) (10 points) Suppose that the particle is initially in a stable circular orbit of radius R_0 and angular momentum L and that at a time t_0 the magnitude of its velocity is instantaneously increased such that the angular momentum is instantaneously increased to a new value $L^{'} > L$ but the instantaneous direction of the velocity is not changed. The new orbit is characterized by a minimum distance from the origin R_{min} and a maximum distance R_{max}. Please explain why $R_{min} = R_0$ and write an equation whose solution gives R_{max} (you are not asked to solve this equation).

After the impulse the particle is moving in the new effective potential

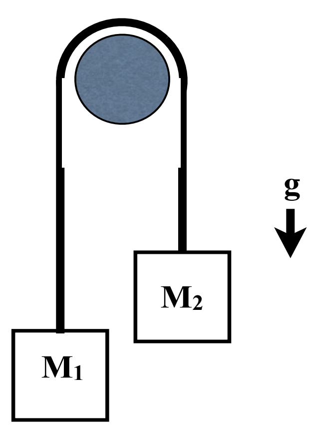
$$U_{new}(r) = \frac{(L')^2}{2\mu r^2} + k\sqrt{r}$$

Immediately after the impulse the particle has $r = R_0$ and $\dot{r} = 0$ so R_0 must be a turning point of the motion in the new effective potential. From the answer to part (c) we see that sin $L' > L$ the minimum of the potential is at a distance greater than R_0 so the turning point must be the minimal value of r. The maximum value or r is then determined from conservation of energy by demanding $U_{new}(R_{max}) = U_{new}(R_{min})$ (because the kinetic energy is the same at both points), in other words

$$\frac{(L')^2}{2\mu R_0^2} + k\sqrt{R_0} = \frac{(L')^2}{2\mu R_{max}^2} + k\sqrt{R_{max}}$$

3 (25 points) The Atwood Spring. Consider two particles of mass m_1 and $m_2$connected by a massless cord which passes over a frictionless pulley (see fig). Suppose the cord is elastic, so it has a natural length L_0 but can be stretched to a length L with energy cost

$$E_{stretch}(L) = 0.5k(L - L_0)^2$$



(a) (10 points) Please write the Lagrangian of the system

Choosing coordinates z_1 and z_2 increasing downwards we have

$$\mathcal{L} = \frac{m_1}{2}\dot{z}_1^2 + \frac{m_2}{2}\dot{z}_2^2 + m_1 g z_1 + m_2 g z_2 - \frac{k}{2}\left(z_1 + z_2 + \pi R - L_0\right)^2$$

(b) (5 points) Please write the Euler-Lagrange equations

$$\begin{aligned} m_1\ddot{z}_1 &= m_1 g - k\left(z_1 + z_2 + \pi R - L_0\right) \\ m_2\ddot{z}_2 &= m_2 g - k\left(z_1 + z_2 + \pi R - L_0\right) \end{aligned}$$

(c) (10 Points) Suppose $m_1 > m_2$ and that at time $t = 0$ the masses are at rest and at the same height. Please solve the equations to find the time dependence of the position of m_1. *Define relative and center of mass coordinates by*

$$\begin{aligned} u_{rel} &= z_1 + z_2 \\ u_{COM} &= \frac{m_1 z_1 - m_2 z_2}{m_1 + m_2} \end{aligned}$$

so that

$$\begin{aligned} z_1 &= \frac{m_2}{m_1 + m_2} u_{rel} + u_{COM} \\ z_2 &= \frac{m_1}{m_1 + m_2} u_{rel} - u_{COM} \end{aligned}$$

so that the Lagrangian becomes

$$\mathcal{L} = \frac{\mu}{2}\dot{u}_{rel}^2 + \frac{M}{2}\dot{u}_{COM}^2 + \delta M g u_{COM} + \mu g u_{rel} - \frac{k}{2}\left(u_{rel} + \pi R - L_0\right)^2$$

with $\mu = m_1 m_2/(m_1 + m_2)$, $M = m_1 + m_2$ and $\delta M = m_1 - m_2$.

The center of mass coordinate thus evolves as

$$U_{COM}(t) = \frac{\delta M g}{2M} t^2 + U_{COM}(t = 0)$$

and the initial condition that $z_1 = z_2 = Z$ then implies

$$u_{COM}(t = 0) = \frac{\delta M}{M} Z$$

Let us rewrite the portion of the Lagrangian for relative coordinate as

$$\mathcal{L}_{rel} = \frac{\mu}{2}\dot{u}_{rel}^2 - \frac{k}{2}\left(u_{rel} - L^\star\right)^2$$

with $L^\star = L_0 - \pi R + \mu g/k$ so that

$$u_{rel}(t) = L^\star + A cos\Omega t$$

with $\Omega = \sqrt{k}/\mu$ and we used the initial condition of zero velocity.

The initial condition that $z_1 = z_2 = Z$ then implies $u_{rel}(t = 0) = 2Z$ so $A = 2Z - L^\star$ and

$$z_1(t) = \frac{\delta M}{M}\left(\frac{g}{2}t^2 + Z\right) + \frac{m_2}{M}\left(L^\star + \left(2Z - L^\star\right) cos\Omega t\right)$$