PHY W3003: Midterm Examination I Solutions
Feb 17, 2016

1 A particle of charge q and mass m moves in three dimensions in the presence of a constant magnetic field directed along z and in the presence of friction which depends linearly on the velocity so

$$\vec{F} = q\vec{v} \times B\hat{z} - \gamma\vec{v}$$

(a) (5 points) Please write the three Newton's equations that determine the three components of the velocity of this system

$$\begin{aligned} m\frac{dv_x}{dt} &= qv_yB - \gamma v_x \\ m\frac{dv_y}{dt} &= -qv_xB - \gamma v_y \\ m\frac{dv_z}{dt} &= -\gamma v_z \end{aligned}$$

(b) (10 points) Please solve the equations to obtain the three components of the velocity at all times $t > 0$ if at time $t = 0$ the particle is moving in the x direction with velocity v_0.

The z component decouples:

$$v_z(t) = v_z^0 e^{-\frac{\gamma}{m}t}$$

The initial condition implies that $v_z^0 = 0$.

For the x and y equations, define $\eta = v_x + iv_y$; then adding the v_x equation to i times the v_y equation and dividing by m gives

$$\frac{d\eta}{dt} = \left(-i\Omega_c - \frac{\gamma}{m}\right)\eta$$

Thus

$$\eta(t) = \eta_0 e^{\left(-i\Omega_c - \frac{\gamma}{m}\right)t}$$

so

$$\begin{aligned} v_x(t) &= v_0 e^{-\frac{\gamma}{m}t} cos\left(\Omega_c t + \phi\right) \\ v_y(t) &= -v_0 e^{-\frac{\gamma}{m}t} sin\left(\Omega_c t + \phi\right) \end{aligned}$$

The initial condition implies $\phi = 0$.

(c) (5 points) Suppose that at time $t = 0$ the particle is at position $\vec{R} = 0$ and has velocity $\vec{v} = v_0\hat{x}$. As time $t \to \infty$ the particle approaches a new position $\vec{R_\infty}$. Please find $\vec{R}_\infty$.

Integrating the solution for $\vec{v}$ gives

$$\begin{aligned} x &= +\int_0^\infty dt v_0 e^{-\frac{\gamma}{m}t} cos\left(\Omega_c t\right) = \frac{\gamma}{\gamma^2 + m^2\Omega_c^2} \\ y &= -\int_0^\infty dt v_0 e^{-\frac{\gamma}{m}t} sin\left(\Omega_c t\right) = -\frac{\Omega}{\gamma^2 + m^2\Omega_c^2} \end{aligned}$$

2 Consider the potential

$$U(x) = U_0\left(1-\frac{x}{a}\right)^2 e^{-\frac{2x}{a}}$$

(a) (5 points) Sketch $U(x)$. Identify the locations of the maxima and minima and be sure that your sketch shows the proper behavior at $x \to \pm\infty$

(b) (10 points) Sketch the phase space orbits, indicating the bound, unbound and separatrix orbits, and give the energy of the separatrices

(c) (10 points) Suppose now that we have a system consisting of two particles: particle 1 with mass m_1 and particle 2 of mass $m_2 = m_1/2$ move in one dimension. Assume that the position x_1 of particle 1 is greater than the position x_2 of particle 2 and suppose that the potential energy of the particles is given by $U(x_1 - x_2)$ with U as above.

(i) Please find the equilibrium separation of the two particles.

The equilibrium position is the minimum of $U(x)$. We have

$$\frac{dU}{dx} = U_0\left(-\frac{2}{a}\left(1-\frac{x}{a}\right) - \frac{2}{a}\left(1-\frac{x}{a}\right)^2\right)e^{-\frac{2x}{a}}$$

So the extrema are at $x = a$ and $x = 2a$ with the minimum (equilibrium position) at $x = a$.

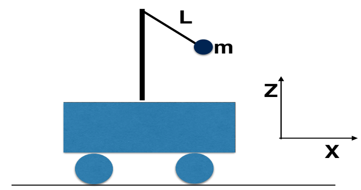
(ii) Suppose that the two particles are initially at rest in their equilibrium positions and that at time $t- = 0$ an impulse is applied so that particle 1 moves with velocity v in the positive x direction. How big must the impulse be (i.e. how big must the velocity of particle 1 be so that at long times the particles become arbitrarily far apart?

the value of the potential at the minimum is 0 and the value at the maximum ($x = 2a$) is U_0e^{-4}.

After the impulse the total momentum is $P = m_1 v$ and the total energy is $E = \frac{m_1 v^2}{2}$. the velocity of the center of mass is $V_{COM} = P_{COM}/(m_1+m_2) = \frac{2v}{3}$. The kinetic energy of the COM motion is then $KE_{COM} = \frac{M_{TOT}V_{COM}^2}{2} = m_1v^2/3$. The kinetic energy of relative motion is thus $KE_{rel} = \frac{m_1v^2}{6}$ and this must be greater than U_0e^{-4} so

$$v > \sqrt{\frac{6U_0e^{-4}}{m_1}}$$

3 A pendulum (mass m attached to a massless rigid rod of length L) rotates in the x-z plane. The suspension point of the pendulum (also massless) is attached to a cart of mass M. The cart rolls without slipping on a horizontal surface as depicted in the Figure. The system is subject to the force of gravity, acting in the vertical direction.



(a) Please write the an expression for the total energy of the system

X is the horizontal position of center of mass of the hoop, Ω is the angular velocity of rotation of the hoop. No slipping condition is $\dot{X} = R\Omega$. The velocity of the mass in the frame of reference of the surface is $R\dot{\phi} + \dot{X}$

$$E = \frac{M\dot{X}^2}{2} + \frac{m\left(R\dot{\phi} + \dot{X}\right)^2}{2} + mgcos\phi$$

For the remainder of the problem please assume that the mass m is released from rest at an angle of 90°. After release it will swing through the vertical direction to an angle of −90°.

(b) At the point when the angle has reached −90° how far is the mass m from its original position? *The center of mass position does not change. The initial position of the COM is*

$$X_{COM}^{init} = \frac{MX^{init} + m(X^{init} + R)}{m + M}$$

The final position of the COM is

$$X_{COM}^{final} = \frac{MX^{final} + m(X^{final} - R)}{m + M}$$

Equating the two and rearranging gives

$$X^{final} - X^{init} = \frac{2mR}{m + M}$$

(c) At the point when the angle 0 (i.e the mass is at the lowest point in its trajectory) what is the velocity of the mass in the reference frame of the surface.

The velocity of the mass is $R\dot{\phi} + \dot{X}$; thus the total momentum, which must be zero, is

$$P_{tot} = m\left(R\dot{\phi} + \dot{X}\right) + M\dot{X} = 0$$

so that

$$\dot{X} = -\frac{m\left(R\dot{\phi} + \dot{X}\right)}{M}$$

and the total kinetic energy is

$$\frac{m^2\left(R\dot{\phi} + \dot{X}\right)^2}{2M} + \frac{m\left(R\dot{\phi} + \dot{X}\right)^2}{2}$$

and we must equate this to $-mgR$ *so*

$$R\dot{\phi} + \dot{X} = \sqrt{\frac{2gR}{1+\frac{m}{M}}}$$

4 A particle of mass m moves in three dimensions subject to the potential

$$V(x, y, z) = A(x^2 + y^2) + Bz^2$$

(a) (10 points) if $A = 0$ and $B > 0$ please state which of the components of the linear momentum $\vec{P}$ and the angular momentum $\vec{L}$ are conserved. *Translation invariance in* x *and* y *but not* z *so* P_x *and* P_y *are conserved. Rotation invariance about* z *but not about* x *or* y *so* L_z *is conserved.*

(b) (10 points) If $A > 0$ and $B = 0$ please state which of the components of the linear momentum $\vec{P}$ and the angular momentum $\vec{L}$ are conserved. *Translation invariance in* z *but not* x *and* y P_z *is conserved. Rotation invariance about* z *but not about* x *or* y *so* L_z *is conserved.*

Please give your reasoning in both cases.