1. Introduction

Consider a particle moving in one dimension. In basic physics courses we represent the motion as the trajectory $X(t)$ and define the velocity as the derivative $V(t) = dX/dt$. In this view the trajectory $X(t)$ is fundamental and the velocity is a derived quantity.

It is useful to take a more abstract view and define a two dimensional phase space with axes $X$ and $V$. At each instant of time the particle has a position and velocity, so we can view the motion of the particle as tracing out a curve in phase space $(X(t), V(t))$ with time as an implicit parameter. It is useful to denote with an arrow the direction of motion along the phase space curve. If we have $M$ particles moving in $N$ dimensions the phase space of the system has dimension $M \times (2N)$. One can guess many of the important features of the motion from phase-plane plots, without solving the equations of motion in detail.

The phase plane concept also provides an important generalization. In mechanics, we specify the state of a particle at time $t$ by its position and velocity at this time. A more general system (for example an embryo undergoing gastrulation) would have its state specified in a different way, but the process by which an embryo develops from a featureless blob of seemingly undifferentiated cells to an organism with different parts can be treated as an evolution in phase space. The subject of “dynamical systems” treats the general properties of this evolution.

2. Examples

2.1. Free particle moving in one dimension. Consider a free particle with coordinate $Z$, obeying Newton’s equation

\begin{equation}
\frac{d^2Z}{dt^2} = 0
\end{equation}

![Position vs time and phase-plane plots](image)

**Figure 1.** Left panel: position $Z$ plotted vs time $t$ from Eq. 2 for $z_0 = 1$ and $v_0 = 1.5$ and $z_0 = 2$ and $v_0 = -0.5$. Right panel: phase-plane plots of same solutions.
Eq. 1 has the solution

\[ Z(t) = v_0 t + z_0 \]  

with \( v_0 \) and \( z_0 \) determined by boundary conditions. The left panel of Fig. 1 shows two trajectories (position \( Z \) plotted vs time) plotted for two different initial conditions. The velocity corresponding to Eq. 2 is

\[ V(t) = v_0 \]

in other words, is constant. Thus the phase-plane plots of the particle motion are very simple: they are just straight lines, parallel to the \( X \) axis. The right panel of Fig. 1 shows the phase plots corresponding to the same two trajectories. Note that for the curve at positive \( V \) the motion is to the right (to larger \( X \)) and for the curve at negative \( V \) the motion is to the left (to smaller \( X \)).

As another example, consider a particle moving in one dimension and subject to constant force. The equation of motion for a particle moving in one dimension and subject to a constant force \( g \) (here directed in the negative direction) is

\[ \frac{d^2 Z}{dt^2} = -g \]

The trajectory is

\[ Z(t) = -\frac{g}{2} t^2 + v_0 t + z_0 \]

and the velocity is

\[ V(t) = -gt + v_0 \]

Solving Eq. 6 to get \( t = (v_0 - V)/g \), substituting into Eq. 5 and simplifying gives

\[ Z = -\frac{V^2}{2g} + \frac{v_0^2}{g} + z_0 \]

so the phase space trajectories are sideways parabolas. Fig. 2 shows two examples computed for \( z_0 = 0 \).
2.1.1. **Linear dissipation.** A particle subject to a linear dissipative (velocity-dependent) force has equation of motion

\[
m \frac{dv}{dt} + bv = 0
\]

with solution

\[
v(t) = v_0 e^{-\gamma t}, \quad x(t) = x_0 + \frac{v_0}{\gamma} (1 - e^{-\gamma t})
\]

Here \( \gamma = b/m, v_0 \) is the initial condition needed to specify the solution of Eq. 8 and the result for \( x \) is obtained by integrating the result for \( v \). Fig. 3 shows the phase-plane plot for Eq. 9 for \( \gamma = 1, x_0 = 0 \) and \( v_0 = 1 \) and 0.9. We see that for this solution the motion is bounded and the line \( v = 0 \) is a fixed line: trajectories simply end on this line—once the particle stops, it does not start moving again.

**3. Energy and Phase Portraits in dimension \( d=1 \)**

One dimensional motion can be very conveniently analysed by a combination of energy and phase-plane plots.

The mechanical energy is

\[
E = \frac{m}{2} v^2 + U(x)
\]

and the requirement that \( v^2 \geq 0 \) implies that particle motion is confined to the regions where \( E > U(x) \).

Nomenclature: regions where \( E < U(x) \) are (classically) forbidden, regions with \( E > U(x_0) \) are (classically) allowed. (I put “classically” in parentheses because the situation is different in quantum mechanics).

Further, for a one dimensional system the energy and position completely determine the velocity. Rearranging Eq. 10 gives

\[
v_E(x) = \pm \sqrt{\frac{2}{m} \sqrt{E - U(x)}}
\]

You should interpret this equation as saying that if we fix the energy, \( E \), then Eq. 11 gives the velocity as a function of position (up to an overall sign). Because \( v \) is a real number, Eq. 11 only makes sense in the (classically) allowed regions.

Start from a point \( x_0 \) in a classically allowed region and increase \( x \). There are two possibilities: either you encounter a point \( x_1 \) at which \( E = U(x_1) \), so that \( v_E(x_1) = 0 \) or as \( x \to \infty \) \( |v| > 0 \) always. In the latter case we say the motion is unbounded: the particle initially at \( x_0 \) and moving in the positive direction will continue in this direction to \( x \to \infty \). In the former case the motion is bounded: the particle initially at \( x_0 \) and moving in the positive direction will slow down as it approaches \( x_1 \) and at \( x_1 \) will have zero velocity. Because in this example the potential must be increasing as \( x \to x_1 \), the force \( dU/dx \) at
$x = x_1$ is in the negative $x$ direction, so there is acceleration in the negative $x$ direction and after reaching $x_1$ the particle will start moving backwards.

Thus to analyse the motion of an energy-conserving one dimensional system you should plot the potential energy vs $x$. For different energies, identify the regions where the motion is bounded in one or both directions, and then below this plot the two branches of Eq. 11. An example is given in Fig. 4.

**Figure 4.** Upper panel: potential $U(x) = A \times x^2/(x^2 + B)^4$ for $A = 10^4$ and $B = 10$. Lower panel: phase plane plots corresponding to energies 0.05, 0.25, 0.65, 1.053 (separatrix), 1.3 and 2 with separatrix indicated by dashed lines.

**Side remark:** The phase space area of a totally bounded orbit has an interesting interpretation. Suppose that we have an orbit confined between $x = x_1$ and $x = x_2$. Then
the phase space area $A$ enclosed by the orbit is

\[(12) \quad A(E) = 2 \int_{x_1}^{x_2} dx v(x) = 2 \sqrt{\frac{2}{m}} \int_{x_1}^{x_2} dx \sqrt{E - U(x)}\]

Thus

\[(13) \quad \frac{dA(E)}{dE} = \sqrt{\frac{2}{m}} \int_{x_1}^{x_2} dx \frac{1}{\sqrt{E - U(x)}} = \frac{2}{m} \int_{x_1}^{x_2} \frac{dx}{v} = \frac{2}{m} \int dt\]

Because the integral is over one half of the orbit, we see that the mass times the derivative of the area of the phase space curve with time gives the period of the orbit.