(1) When does a skier leave the slope?

Consider a particle subject to the force of gravity moving on a curve \( z = h(x) \) (think of a skiier sliding down a mountain) so the Lagrangian is

\[
L = \frac{m}{2} (\dot{x}^2 + \dot{z}^2) - mgz + \lambda (z - h(x))
\]

The particle will stay on the curve if the force of constraint is such as to oppose the force of gravity but will leave the curve if the force changes sign (the slope can push you up but not down).

Suppose at time \( t = 0 \) the particle is at rest at \( x = 0 \) and height \( z_0 = h(0) \)

(a) Write the Euler-Lagrange equations including the constraint equation. Note that these will involve derivatives of \( h \).

\[
\begin{align*}
    m\ddot{x} &= -\lambda h' \\
    m\ddot{z} &= -mg + \lambda \\
    0 &= z - h(x)
\end{align*}
\]

\((h' = dh/dx)\)

(b) From the Euler-Lagrange equation for \( z \) show that

\[
\lambda = mg + m\frac{dh}{dx} \ddot{x} + m\frac{d^2 h}{dx^2} \dot{x}^2
\]

\((hint: differentiate the constraint equation to find an equation for \( \ddot{z} \) in terms of derivatives of \( h \) and \( x \))

From the constraint equation,

\[
\begin{align*}
    \dot{z} &= h' \dot{x} \\
    \ddot{z} &= h'' \dot{x}^2 + h' \ddot{x}
\end{align*}
\]

so from the \( z \) equation

\[
\lambda = mg + mh' \ddot{x} + mh'' \dot{x}^2
\]
and then show that the equation of motion for $x$ can be written
\[
\left(1 + \left(\frac{dh}{dx}\right)^2\right) \ddot{x} + \frac{dh}{dx} \frac{d^2h}{dx^2} \dot{x}^2 + g \frac{dh}{dx} = 0
\]

Then substituting into the $x$ equation gives
\[
m \ddot{x} = -mg' - m (h')^2 \dot{x} - m h'' h' \dot{x}^2
\]
so bringing everything to the left side and dividing by $m$ gives the needed result.

(c) By using the equation of motion for $x$ to eliminate $\ddot{x}$ and conservation of energy to determine $\dot{x}^2$ show that the condition for the skier to leave the slope is
\[
1 + \left(\frac{dh}{dx}\right)^2 + 2 (z_0 - h) \frac{d^2h}{dx^2} = 0
\]

The condition that the skier leave the slope is that constraint force $\lambda$ on the skier changes sign (the slope cannot supply a negative constraint force).

To obtain $\lambda$ we note that the equation derived in part (b) implies
\[
\ddot{x} = -h' \frac{h''}{1 + (h')^2} \dot{x}^2 + g
\]
Substituting into the equation for $\lambda$ gives
\[
\frac{\lambda}{m} = g - (h')^2 \frac{h''}{1 + (h')^2} \dot{x}^2 + h'' \dot{x}^2 = \frac{h''}{1 + (h')^2} \dot{x}^2 + g
\]
and noting that conservation of energy relates the kinetic energy to the vertical distance fallen as $\frac{1}{2}mv^2 = mg(z_0 - h(x))$ so the magnitude of the velocity $v^2 = 2(z_0 - h(x))$. The $x$ component $\dot{x} = v/\sqrt{1 + (h')^2}$ so
\[
\frac{\lambda}{m} = g \frac{(z_0 - h(x))}{1 + (h')^2} + 1
\]
The numerator vanishes when the stated condition is satisfied.

If you are feeling ambitious you may also show that this criterion is simply the point at which the local centripetal force equals the component of the gravitational force normal to the curve.

The component of the gravitational force normal to the track is
\[
g_{\text{Norm}} = -\frac{mg}{\sqrt{1 + (h')^2}}
\]
The centripetal force is $mv^2/R$ where $R$, the local radius of curvature, is given by
\[
R = \frac{\left(1 + (h')^2\right)^{\frac{3}{2}}}{h''}
\]
so
\[ F_{\text{centrip}} = \frac{m\dot{x}^2}{\left(1 + (\phi')^2\right)^{\frac{3}{2}}} \]
equating gives the condition as \( h''\dot{x}^2 = -g \) which is a version of what we had before.

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The potential energy is \( V = T (d_1 + d_2 + d_3) \) where the change in lengths of the spring are
\[
d_1 = \sqrt{L^2 + y_1^2} - L = \frac{y_1^2}{2}
\]
\[
d_2 = \sqrt{L^2 + (y_1 - y_2)^2} - L = \frac{(y_2 - y_1)^2}{2}
\]
\[
d_3 = \sqrt{L^2 + y_2^2} - L = \frac{y_2^2}{2}
\]
so the Lagrangian is
\[
\mathcal{L} = \frac{m}{2} (\dot{y}_1^2 + \dot{y}_2^2) - \frac{T}{2} (2y_1^2 + 2y_2^2 - 2y_2y_1)
\]
Thus if we express our coordinates \( y_1 \) and \( y_2 \) as a two component vector \( \vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \)
then the E-L equations are
\[
\mathbf{M}\ddot{\vec{y}} = -\mathbf{K}\vec{y}
\]
with
\[
\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]
\[
\mathbf{K} = T \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}
\]
The mass matrix is proportional to the unit matrix so is diagonal in any basis; the force matrix has eigenvectors \( \eta_1 = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \) (eigenvalue \( T \)) and \( \eta_2 = \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} \) (eigenvalue \( 3T \)). So the motion is the superposition of two oscillations: a uniform translation in which both particles move up and down together (frequency \( \omega_1 = \sqrt{\frac{T}{m}} \)) and a “beating” where they move opposite (frequency \( \omega = \sqrt{\frac{3T}{m}} \)).

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The position of the cart is \( x \), the position of the pendulum is \( x_p = x + L\sin\phi \), \( z_p = -L\cos\phi \); thus the kinetic energy is
\[
KE = \frac{m}{2} \dot{x}^2 + \frac{M}{2} \left( \dot{x} + L\cos\phi \dot{\phi} \right)^2 + L^2 \sin^2\phi \dot{\phi}^2 \to \frac{m}{2} \dot{x}^2 + \frac{M}{2} \left( \dot{x} + L\dot{\phi} \right)^2
\]
where the arrow indicates the result of the small angle approximation. Similarly the potential energy is \(Mgz = -MgL \cos \phi \to \frac{MgL\phi^2}{2}\).

Thus the Euler-Lagrange equations are

\[
\begin{align*}
(m + M)\ddot{x} + M \dot{L} \dot{\phi} &= 0 \\
ML\ddot{x} + M \dot{L}^2 \dot{\phi} &= -MgL\phi
\end{align*}
\]

It is easiest to solve this by hand. Translational invariance implies a conserved momentum so there is a zero mode (mode frequency zero), which is \(\phi = 0\), \(x(t) = x_0 + vt\). To find the other mode we eliminate \(\ddot{x}\) using the first equation to obtain

\[
\left( -\frac{M^2 L^2}{m + M} + M \dot{L}^2 \right) \ddot{\phi} = -MgL\phi
\]

or, tidying up the equation

\[
\frac{m}{m + M} \ddot{\phi} = -\frac{g}{L} \phi
\]

so we have a normal mode of frequency \(\omega = \sqrt{\frac{g(m+M)}{mL}}\) in which the pendulum and the cart oscillate out of phase in such a way that the center of mass momentum stays fixed.

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We have

\[
\mathcal{L} = \frac{m}{2} \dot{x}_1^2 + \frac{M}{2} \dot{x}_1^2 + \frac{m}{2} \dot{x}_1^2 - \frac{k}{2} \left( (x_2 - x_1)^2 + (x_3 - x_2)^2 \right)
\]

so writing the normal coordinates as a three component vector \(\vec{q} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\) the E-L equations are

\[
\mathbf{M} \ddot{\vec{q}} = -\mathbf{K} \vec{q}
\]

with

\[
\mathbf{M} = \begin{pmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{pmatrix}
\]

\[
\mathbf{K} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}
\]

It is instructive to solve the equations in the informal way or the formal way. Start with the informal way.
We know there are three modes. The problem has a translation invariance, so moving all of the particles by the same amount must be a zero model. Indeed
\[ \vec{q}_{\text{trans}} = \begin{pmatrix} q \\ q \\ q \end{pmatrix} \] is an eigenvector of \( K \) with eigenvalue zero.

The other two modes must keep the COM fixed. One way to do this is to keep particle 2 fixed, and move particles 1 and 3 in opposite directions. This suggests we try \( \vec{q}_{\text{breathe}} = \begin{pmatrix} q \\ 0 \\ -q \end{pmatrix} \). Substitution shows that \( \vec{q}_{\text{breathe}} \) is indeed an eigenvector of \( M \) with eigenvalue \( m \) and an eigenvector of \( K \) with eigenvalue \( k \) so \( \vec{q}_{\text{breathe}} \) is a normal mode with frequency \( \omega_{\text{breathe}} = \sqrt{\frac{k}{m}} \).

Another way to keep the COM fixed is to move mass 2 and move masses 1 and 3 by equal amounts in the opposite direction to keep \( m(x_1 + x_3) + Mx_2 \) fixed; in other words to say \( x_1 = x_3 = -\frac{M}{2m}x_2 \) so our vector is \( \vec{q}_{\text{beat}} = \begin{pmatrix} q(t) \\ -2q(t) \frac{m}{M} \\ q(t) \end{pmatrix} \).

We see that
\[
\begin{align*}
M\dddot{\vec{q}}_{\text{beat}} &= m\dddot{q} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \\
K\dddot{\vec{q}}_{\text{beat}} &= -kq \begin{pmatrix} 1 + \frac{2m}{M} \\ -2 \\ 1 \end{pmatrix}
\end{align*}
\]

Thus while \( \vec{q}_{\text{beat}} \) is not an eigenvector of \( M \) or \( K \) individually, it is a solution of the equation if \( q(t) \) has harmonic time dependence with frequency \( \omega_{\text{beat}} = \sqrt{\frac{k(2m+M)}{mM}} \). Note that as \( M \to \infty \) the motion of the central atom becomes negligible and the mode just involves the two outer atoms moving in the same direction and the mode frequency only involves the light mass; similarly if \( M \ll m \) then the only motion is of the inner atom with frequency \( \sqrt{2k/M} \) as expected.

Now the formal way.
Write \( \vec{q} = M^{-\frac{1}{2}} \vec{\eta} \) with
\[
M^{-\frac{1}{2}} = \begin{pmatrix} \frac{1}{\sqrt{m}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{M}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{m}} \end{pmatrix}
\]
and multiply the equation from the left by \( M^{-\frac{1}{2}} \) obtaining
\[ \dddot{\vec{\eta}} = -K'\vec{\eta} \]
with
\[ K' = M^{-\frac{1}{2}}KM^{-\frac{1}{2}} = \begin{pmatrix} \frac{k}{m} & \frac{k}{\sqrt{mM}} & 0 \\ \frac{k}{m} & \frac{2k}{\sqrt{mM}} & \frac{k}{2\sqrt{mM}} \\ 0 & \frac{k}{\sqrt{mM}} & \frac{k}{m} \end{pmatrix} \]

This is a real symmetric matrix. It has three orthogonal eigenvectors. We can find one by inspection:

\[ \eta_{\text{breathe}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \quad \omega_{\text{breathe}} = \sqrt{\frac{k}{m}} \]

The other two must be orthogonal to \( \eta_{\text{breathe}} \). Two possibilities are

\[ \eta_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \eta_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \]

Now we have

\[ K'\vec{\eta}_1 = \begin{pmatrix} \sqrt{\frac{2k}{mM}} \\ 2k \sqrt{\frac{k}{mM}} + \sqrt{2} \frac{k}{\sqrt{mM}} \end{pmatrix} = \sqrt{2} \frac{k}{\sqrt{mM}} \eta_1 + \sqrt{2} \frac{k}{\sqrt{mM}} \eta_2 \]

and

\[ K'\vec{\eta}_2 = \begin{pmatrix} \frac{k}{\sqrt{2m}} \\ \frac{k}{\sqrt{2m}} \end{pmatrix} = \sqrt{2} \frac{k}{\sqrt{mM}} \eta_1 + \frac{k}{m} \eta_2 \]

so in this subspace we can represent the action of \( K' \) as a \( 2 \times 2 \) matrix \( K'' \)

\[ K'' = \begin{pmatrix} \frac{2k}{mM} & \sqrt{2} \frac{k}{\sqrt{mM}} \\ \sqrt{2} \frac{k}{\sqrt{mM}} & \frac{k}{m} \end{pmatrix} \]

with eigenvalue equation

\[ \lambda^2 - \left( \frac{2k}{M} + \frac{k}{m} \right) \lambda = 0 \]

(note that we see that the determinant of the matrix is zero so one of the eigenvalues is zero. The other is

\[ \lambda = \frac{k(2m + M)}{mM} \]

which is what we had before.

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NOTE!!! It is crucial here that the three masses are equal so the mass matrix is proportional to the unit vector. The normal modes are otherwise orthogonal only in terms of the metric defined by the mass matrix!!
The three normal mode vectors are

\[ \eta_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \eta_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad \eta_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \]

These three vectors are mutually perpendicular. Thus, given any vector \( \vec{v} \) we can write

\[ \vec{v} = n_1 \eta_1 + n_2 \eta_2 + n_3 \eta_3 \]

with the coefficients obtained by taking the dot product of our target vector with each of the basis vectors in turn: \( n_i = \vec{v} \cdot \vec{\eta}_i \).

On a more pedestrian level, straightforward algebra shows

\[
\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{3}} \eta_1 + \frac{1}{\sqrt{2}} \eta_2 + \frac{1}{\sqrt{6}} \eta_3 \\
\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{3}} \eta_1 - \frac{2}{\sqrt{6}} \eta_3 \\
\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{3}} \eta_1 - \frac{1}{\sqrt{2}} \eta_2 + \frac{1}{\sqrt{6}} \eta_3
\]

so

\[
\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \frac{v_1 + v_2 + v_3}{\sqrt{3}} \eta_1 + \frac{v_1 - v_3}{\sqrt{2}} \eta_2 + \frac{v_1 - 2v_2 + v_3}{\sqrt{6}} \eta_3
\]

The expansion coefficients are the normal coordinates; they give the relative displacements of the three pendulum bobs in a given normal mode.