(1) Counts as two problems Consider a general harmonically driven oscillator obeying the equation

$$m\frac{d^2 x}{dt^2} + \frac{2m}{\tau} \frac{dx}{dt} + kx = mf_d \cos(\Omega_d t)$$

(a) Find $x(t > 0)$ and $v(t > 0)$ assuming that both $x$ and $v = dx/dt$ equal 0 at $t = 0$. The idea here is to choose the solution of the homogeneous equation to exactly cancel the particular solution at $T = 0$. It is easiest to write a general solution to the complex equation. The general solution of the homogeneous equation is

$$\eta_h(t) = \eta_+ e^{z_+ t} + \eta_- e^{z_- t}$$

with

$$z_\pm = \frac{1}{\tau} \pm \sqrt{\frac{1}{\tau^2} - \Omega_0^2}$$

and $\omega_0^2 = \frac{k}{m}$. The particular solution is

$$\eta_p(t) = \frac{f_d e^{i\Omega_d t}}{\omega_0^2 - \Omega_d^2 + \frac{2\Omega_d}{\tau}}$$

Thus we require

$$\eta_+ + \eta_- = \frac{f_d}{\omega_0^2 - \Omega_d^2 + \frac{2\Omega_d}{\tau}}$$

$$z_+ \eta_+ + z_- \eta_- = \frac{if_d \Omega_d}{\omega_0^2 - \Omega_d^2 + \frac{2\Omega_d}{\tau}}$$

so

$$\eta_+ = \frac{f_d (i\Omega_d - z_-)}{(z_+ - z_-) \left(\omega_0^2 - \Omega_d^2 + \frac{2\Omega_d}{\tau}\right)}$$

$$\eta_- = \frac{f_d (-i\Omega_d + z_+)}{(z_+ - z_-) \left(\omega_0^2 - \Omega_d^2 + \frac{2\Omega_d}{\tau}\right)}$$

We saw in class that the steady state solution (at long times, after the transients have died away) is of the form

$$x(t) = R f_d \cos(\Omega_d t - \phi)$$

with $R$ and $\phi$ determined by $\tau$ and $k/m$.\[1\]
(b) In the long time steady state (after the transients have died away) suppose that over one quarter period, \( x \) changes from 0 to \( R_f \). Please compute the energy put in to the oscillator over this time by the driving force (work done by the driving force) and the energy taken out of the oscillator by friction (work done by friction). The work is given by \( \int Fv \, dt \). The long-time steady state solution is \( x(t) = R_f \cos(\Omega_d t - \delta) \) so the velocity is \( -R_f \omega_d \sin(\Omega_d t - \delta) \). \( x = 0 \) when \( t = (\delta - \frac{\pi}{2})/\Omega_d \) and takes its maximum value when \( t = \delta/\Omega_d \) so we have

\[
W_{\text{app force}} = \int_{\delta - \frac{\pi}{2}\Omega_d}^{\delta\Omega_d} dt f_d \cos(\Omega_d t) \left( R_f \Omega_d \sin(\Omega_d t - \delta) \right) = \frac{R_f^2}{4} \left( \pi \sin\delta + 2 \cos\delta \right)
\]

\[
W_{\text{damping}} = \int_{\delta - \frac{\pi}{2}\Omega_d}^{\delta\Omega_d} dt \frac{-2}{\tau} \left( R_f \Omega_d \sin(\Omega_d t - \delta) \right)^2 = -\frac{\pi R^2 f_d^2 \Omega_d}{2\tau}
\]

Consider the special case \( k/m = 1, \Omega_d = 1 \) and \( \tau = 1 \). This oscillator is critically damped and the drive frequency is equal to the natural frequency. The general solution of the homogeneous equation is

\[ x(t) = Ae^{-\frac{t}{\tau}} + Bte^{-\frac{t}{\tau}} \]

The particular solution is

\[ x_p(t) = \frac{f_d}{2} \sin(t) \]

(c) Find explicit expressions for \( x(t > 0) \) and \( v(t > 0) \) assuming that both \( x \) and \( v = dx/dt \) equal 0 at \( t = 0 \). The initial position of the particular solution is zero and the velocity is \( f_d/2 \) so \( A = 0 \) and \( B = -f_d/2 \) so

\[ x(t) = \frac{f_d}{2} \left( \sin(t) - te^{-t} \right) \]

\[ v(t) = \frac{f_d}{2} \left( \cos(t) - e^{-t} + te^{-t} \right) \]

(d) Find \( x(t > 0) \) and \( v(t > 0) \) assuming that at \( t = 0 \), \( x = 0 \) but \( v = 0.5 \).

Still \( A = 0 \) and \( B = -f_d/2 + 0.5 \) so

\[ x(t) = \frac{f_d}{2} \sin(t) - \left( \frac{f_d}{2} - \frac{1}{2} \right) te^{-t} \]

\[ v(t) = \frac{f_d}{2} \cos(t) - \left( \frac{f_d}{2} - \frac{1}{2} \right) \left( 1 - t \right) e^{-t} \]

(e) Sketch (on one graph) the phase plane portraits showing the evolution of these two solutions for \( t > 0 \).

(2) 5.43 (variant): answer this question but instead of using the data in the book assume that the car sinks by 2 cm if one 80kG passenger climbs in and that the axle assembly has total mass 10kG.
Here we need to estimate the spring constant for the wheel assembly and then note that the spring constant, plus the mass of the wheel assembly, implies an oscillation frequency for the car.

(a) Car without passenger has mass \(M\) and is supported by the 2 axle assemblies. Each axle assembly has 2 springs, each of spring constant \(K\). Thus the extension of the springs \(H\) is given by

\[
4KH = gM
\]

We add a person, mass 80kG and \(H \rightarrow H + 0.02m\). Thus approximating \(g = 10m/sec^2\)

\[
4K \times 0.02m = g(80kG)
\]

so

\[
K = \frac{80 \times 10}{4 \times 0.02} = 10 \times 10^3 N/m^2
\]

(b) Because each axle has 2 wheels the total spring constant for the axle is \(2K = 2 \times 10^4 N/m^2\). The mass of the axle assembly is 20kG so the oscillation frequency \(\omega_0\) is

\[
\omega_0 = \sqrt{\frac{2K}{m_{axle}}} = \sqrt{\frac{2 \times 10^4}{20}} \approx 10/sec
\]

This implies that the period \(2\pi/\omega_0 \approx 0.6\) sec. If the car is driven over a road with bumps spaced 0.8m apart then the velocity at which these bumps will resonate with the suspension of the car is \(0.8/0.6 \approx 1.3m/sec\)
(3) (counts as two problems)

(a) Find the Fourier series for a periodic square wave of period $T_0$ defined by

$$f(t) = \begin{cases} 
+3 & \text{for } 0 < t < \frac{T_0}{8}, \text{ and } \frac{7T_0}{8} < t < T_0 \\
-1 & \text{for } \frac{T_0}{8} < t < \frac{7T_0}{8},
\end{cases}$$

with $\omega_n = \frac{2\pi n}{T_0}$ and

$$f_n = \int_{0}^{T_0} dt e^{-i\omega_n t} f(t) = \frac{T_0}{2\pi i n} \left(3 \left( e^{\frac{2\pi i n}{8}} - 1 \right) - \left( e^{\frac{14\pi i n}{8}} - e^{\frac{2\pi i n}{8}} \right) + 3 \left(1 - e^{\frac{14\pi i n}{8}} \right) \right)$$

$$= \frac{4T_0}{2\pi i n} \left( e^{\frac{\pi i n}{4}} - e^{\frac{7\pi i n}{4}} \right) = \frac{4T_0}{\pi n} (-1)^n \sin \frac{3\pi n}{4}$$

(b) Use the result in (a) to find the steady state motion of a damped harmonic oscillator that is driven by the force $F_d f(t)$ with $f(t)$ defined in (a) and amplitude $F_d$ arbitrary. In particular, find the relative amplitudes $A_1, A_3,$ and $A_5$ of the first three terms of the response function $x(t)$ in the case that the third harmonic $3\omega$ of the driving frequency coincides with the frequency $\omega_0$ of the undamped oscillator. Let the quality factor $Q = 100$.

The particular solution is

$$x(t) = \frac{F_d}{mT_0} \text{Re} \left[ \sum_n f_n e^{i\omega_n t} \right]$$

In the particular case specified here $\omega_0 = \frac{6\pi}{T_0}$ and $Q = \omega_0 \tau/2$, i.e. $\frac{\tau}{\tau} = \frac{\omega_0}{2Q}$ so the $n^{th}$ term in the response function is

$$R_n = \frac{4F_d}{m} \frac{(-1)^n \sin \frac{3\pi n}{4}}{\pi n \left( \frac{4\pi^2}{T_0^2} (9 - n^2) + \frac{12\pi^2 n}{Q^2 T_0^2} \right)}$$

so the magnitudes, taking out all common factors are, for $n = 1, 3, 5$

$$A_{n=1,3,5} = \frac{1}{n} \frac{1}{\sqrt{(9 - n^2)^2 + \frac{9n^2}{Q^2}}}$$

so

$$\frac{A_3}{A_1} = \frac{\sqrt{64.009}}{0.081}$$

$$\frac{A_5}{A_1} = \frac{\sqrt{64.009}}{\sqrt{256.225}}$$
(4) 5.30 The general solution is

\[ x(t) = C_+ e^{-\frac{t}{\tau_+}} + C_- e^{-\frac{t}{\tau_-}} \]

implying

\[ v(t) = -\frac{C_+}{\tau_+} e^{-\frac{t}{\tau_+}} - \frac{C_-}{\tau_-} e^{-\frac{t}{\tau_-}} \]

Thus

\[ x(t = 0) = C_+ + C_- \]
\[ v(t = 0) = -\frac{C_+}{\tau_+} - \frac{C_-}{\tau_-} \]

so

\[ C_+ = \frac{\tau_+ x(t = 0) + \tau_+ \tau_- v(t = 0)}{\tau_+ - \tau_-} \]
\[ C_- = -\frac{\tau_- x(t = 0) + \tau_+ \tau_- v(t = 0)}{\tau_+ - \tau_-} \]

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(b) \( v(t = 0) = 0 \) means

\[ C_+ = \frac{\tau_+ x(t = 0)}{\tau_+ - \tau_-} \quad C_- = -\frac{\tau_- x(t = 0)}{\tau_+ - \tau_-} \]

\( x(t = 0) = 0 \) means

\[ C_+ = \frac{\tau_+ \tau_-}{\tau_+ - \tau_-} v(t = 0) \quad C_- = -\frac{\tau_+ \tau_-}{\tau_+ - \tau_-} v(t = 0) \]

Figure 1. Position (solid line) and velocity (dashed line) for overdamped oscillator with \( \tau_{\pm} = (1.5, .5) \)

(c) As \( \tau \to \infty \) we have

\[ \frac{1}{\tau_{\pm}} \to \pm \sqrt{-\omega^2} = \pm i \omega \]