PHY 3003 SPRING 2014

(1) ‘Tetherball’: consider a point mass $m$ suspended from a point by a rigid massless rod of length $D$ (so that it is free to rotate over the surface of a sphere) and subject to the force of gravity. Write the position of the mass $\vec{r} = (x, y, z) = D(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ and take gravity along $z$.

Here you should think of $\theta$ as being analogous to the radial coordinate in the two body central force problem.

In terms of the angular variables the velocities are

$$v_x = D\cos\theta \cos\phi - D\sin\theta \sin\phi \dot{\phi}$$
$$v_y = D\cos\theta \sin\phi + D\sin\theta \cos\phi \dot{\phi}$$
$$v_z = -D\sin\theta \dot{\theta}$$

Thus

$$\vec{v}^2 = D^2 \dot{\theta}^2 + D^2 \sin^2\theta \dot{\phi}^2$$

while

$$L_z = xmv_y - ymv_x = mD^2 \sin^2\theta \dot{\phi}$$

(a) Use conservation of the $z$-component of angular momentum to eliminate $d\phi/dt$ and obtain an expression for the energy in terms of $\theta$ and $d\theta/dt$.

The energy is (with $\theta = 0$ vertically down)

$$E = \frac{m}{2} \vec{v}^2 - mgD\cos\theta = \frac{mD^2}{2} \left( \dot{\theta}^2 + \sin^2\theta \dot{\phi}^2 \right) - mgD\cos\theta$$

so using our Eq. for $L_z$

$$E = \frac{mD^2}{2} \dot{\theta}^2 + \frac{L_z^2}{2mD^2 \sin^2\theta} - mgD\cos\theta$$

(b) Show that if $L \neq 0$ and the energy is fixed, then no matter how large the energy is, the motion is confined between an angle $\theta_0 > 0$ and an angle $\theta_\pi > \pi$, and explain physically why $\theta_0 > \pi - \theta_\pi$

The effective potential $\frac{L_z^2}{2mD^2 \sin^2\theta} - mgD\cos\theta$ diverges both as $\theta \to 0$ and as $\theta \to \pi$ because $\sin\theta \to 0$ at these points.

At the points $\theta = \theta_0$ and $\theta = \theta_\pi$ the energy equals the effective potential so $\dot{\theta} = 0$ and the energy is the combination of potential energy $mg\cos\theta$ and rotation around $z$ ($mD^2 \sin^2\theta \dot{\phi}^2 = \frac{L_z^2}{2mD^2 \sin^2\theta}$). At $\theta = \theta_0$ the potential energy is more negative than at $\theta = \theta_\pi$ so the kinetic energy of rotation is higher. Because $L_z$ is fixed this means $\sin^2\theta_0 < \sin^2\theta_\pi = \sin^2(\pi - \theta_\pi)$.
For a given value of angular momentum $L$, a stable motion is possible in which $\theta = \theta_L$ is independent of time. Please find $L$ in terms of $\theta_L$, $m$, $g$ and $D$.

Stable motion is possible if the angle is the angle that minimizes the effective potential and the energy is equal to the value at the minimum. Minimizing the effective potential with respect to $\theta$ we get

$$\frac{dU_{\text{eff}}}{d\theta} = -\frac{L_z^2}{mD^2\sin^3\theta} + mgD\sin\theta$$

Equating this to zero and solving for $L_z$ gives

$$L_z = mD\frac{3}{2}g\frac{1}{2}\sin^2\theta$$

(2) Solving a trivial problem the hard way. Suppose that we have two particles of masses $m_1$ and $m_2$ moving in two dimensions, and suppose that the particles do not interact with each other or with anything else, so they move in straight lines. Assume further that the particles move in opposite directions with the distance of closest approach being $b$. A solution of Newton’s equations satisfying these conditions is

$$\vec{r}_1(t) = vt\hat{x} + \frac{b}{2}\hat{y}$$
$$\vec{r}_2(t) = -vt\hat{x} - \frac{b}{2}\hat{y}$$

(a) Please find the time dependence of the center of mass position $\vec{R}_{\text{COM}}$

$$\vec{R}_{\text{COM}}(t) = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} = \frac{m_1 - m_2}{m_1 + m_2} \left( vt\hat{x} + \frac{b}{2}\hat{y} \right)$$

(b) Find the time dependence of the relative coordinate $\vec{r} = \vec{r}_1 - \vec{r}_2$

$$\vec{r}(t) = 2vt\hat{x} + b\hat{y}$$

(c) Find the value $z$ component of the angular momentum measured relative to the center of mass position.

Expressing the particle positions relative to the center of mass position in terms of relative and center of mass coordinates gives

$$\vec{r}_1 - \vec{R} = \frac{m_2}{m_1 + m_2} \vec{r}$$
$$\vec{r}_2 - \vec{R} = -\frac{m_1}{m_1 + m_2} \vec{r}$$
so that writing \((\vec{r}_1 - \vec{R}) \times (m_1 \vec{v}_1 - m_1 \vec{V}) + (\vec{r}_2 - \vec{R}) \times (m_2 \vec{v}_2 - m_2 \vec{V})\) and using the definition of the center of mass position to cancel the cross terms gives
\[
L = \mu \vec{r} \times \vec{\dot{r}}
\]

Evaluating explicitly gives
\[
L = -2\mu bv
\]

(d) By expressing \(\vec{r}\) in polar coordinates \(r, \theta\), find the time dependence of \(r\) and \(\theta\). Give the values of \(\theta\) for time \(t \to -\infty\) and \(t \to +\infty\),
\[
r = \sqrt{4v^2t^2 + b^2} \quad \theta = \frac{\pi}{2} - \arctan \frac{2vt}{b}
\]

Thus at \(t \to -\infty\) \(\theta = \pi\) and as \(t \to +\infty\), \(\theta \to 0\).

(e) Show that the energy of the system may be expressed in terms of the \(z\) component of angular momentum measured relative to the center of mass (call this \(L\)), the time derivative of the radial polar coordinate \(r\), and the kinetic energy associated with the COM motion as
\[
E = \frac{m_1 + m_2}{2} \frac{V^2}{\text{COM}} + \frac{\mu}{2} \dot{r}^2 + \frac{L^2}{2\mu r^2}
\]

\((\mu\) is the reduced mass)

Write
\[
E = \frac{m_1 v_1^2}{2} + \frac{m_2 v_2^2}{2}
\]

and use the equations relating \(r_{1,2}\) to \(R\) and \(r\). This gives
\[
E = \frac{m_1 + m_2}{2} V^2 + \frac{m_1 m_2}{2(m_1 + m_2)} \dot{r}^2
\]

Now expressing the time derivatives of \(\vec{r}\) in terms of the time derivatives of \(r\) and \(\theta\) and using the definition of \(L\) gives the desired answer

(f) The energy of the system is of course \(\frac{1}{2}(m_1 + m_2) v^2\). Show that the value of the energy plus the angular momentum you found in (d) implies that the smallest possible value of \(r\) is \(b\).

At the point of closest approach \(\dot{r} = 0\) and \(r = b\) so
\[
E = \frac{m_1 + m_2}{2} \frac{V^2}{\text{COM}} + \frac{L^2}{2\mu b^2}
\]

Using our result for \(L = -2bv\mu\) and for \(V = \frac{m_1 - m_2}{m_1 + m_2} v\) gives
\[
E = \frac{(m_1 - m_2)^2}{2(m_1 + m_2)} v^2 + 2\mu v^2 = \frac{1}{2} (m_1 + m_2) v^2
\]

(g) From the expression
\[
\dot{r} = \pm \sqrt{\frac{2}{\mu} \left( E - E_{\text{COM}} - \frac{L^2}{2\mu r^2} \right)}
\]
obtain an expression for $r(t > 0)$ given that at $t = 0$ $r = b$ (you should be able to
do the integral!)

From our result for the energy we have (considering that for $t > 0$ $r$ increases)

$$\dot{r} = \sqrt{\frac{L^2}{\mu^2} - \frac{L^2}{\mu^2 r^2}} = \frac{L}{\mu b} \sqrt{1 - \frac{b^2}{r^2}}$$

Thus

$$\frac{\mu b}{|L|} \frac{dr}{\sqrt{1 - \frac{b^2}{r^2}}} = dt$$

so (noting that $L = -2\mu v b$) we have

$$t = \int_{b}^{r} \frac{dr'}{2v \sqrt{1 - \frac{b^2}{(r')^2}}} = \int_{b}^{r} \frac{r'dr'}{2\sqrt{(r')^2 - b^2}} = \frac{1}{2v} \sqrt{r^2 - b^2}$$

so that

$$4v^2 t^2 + b^2 = r^2$$

(h) From the expression $L = \mu r^2 \dot{\theta}$ and your knowledge of $r(t)$ obtain an expression for $\theta(t)$ and show that it agrees with what you found in (d). In particular,
what is the change in $\theta$ as time goes from 0 to $\infty$? (Hint: you may find it easier to
write $d\theta/dt = d\theta/dr/(dr/dt)$, integrate this to obtain an explicit integral for $\theta$ as a
function of $r$ and then use your result for $r(t)$).

$$\dot{\theta} = \frac{L}{\mu r^2} = \frac{-2\mu v b}{\mu (4v^2 t^2 + b^2)}$$

so

$$\theta(t) - \theta(0) = \int_{0}^{t} \frac{-2vbdt'}{(4v^2(t')^2 + b^2)} = -\int_{0}^{t} \frac{b}{2v} \frac{dt'}{(t')^2 + \frac{b^2}{4v^2}} = -\text{ArcTan} \frac{2vt}{b}$$