(1) In this problem we investigate the Lagrangian formulation of dynamics in a rotating frame. Consider a frame of reference which we will consider to be inertial. Suppose that in this frame of reference we have a particle, of mass $m$, with position vector $\vec{R}$. Now consider a second reference frame, rotating with respect to the first, with rotation vector $\vec{\Omega}$. Let the position of the particle in this reference frame be $\vec{R}$. 

(a) Show that if the Lagrangian in the un-rotating frame is

$$L = \frac{m}{2} \dot{\vec{R}}^2 - V(\vec{R})$$

then in the rotating frame

$$L = \frac{m}{2} \dot{\vec{R}}^2 + m \dot{\vec{R}} \cdot (\vec{\Omega} \times \vec{R}) + \frac{m}{2} (\vec{\Omega} \times \vec{R})^2 - V(\vec{R})$$

The rule for derivatives in a rotating frame is

$$\dot{\vec{R}} \rightarrow \dot{\vec{R}} + \vec{\Omega} \times \vec{R}$$

Making this substitution in $L$ gives

$$L = \frac{m}{2} (\dot{\vec{R}} + \vec{\Omega} \times \vec{R})^2 - V(\vec{R})$$

and expanding the square gives the desired answer.

(b) Show that the Euler-Lagrange equation corresponding to this Lagrangian is

$$m \ddot{\vec{R}} = 2m \dot{\vec{R}} \times \vec{\Omega} + m (\vec{\Omega} \times \vec{\Omega}) \times \vec{R} - \vec{\nabla}_R V$$

Be careful about the order of the factors in the cross product. Note that the $m \dot{\vec{R}} \cdot (\vec{\Omega} \times \vec{R})$ term contributes both to the derivative of $L$ with respect to $\dot{\vec{R}}$ and with respect to $\vec{R}$, and use the triple product rule $\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{C} \cdot (\vec{A} \times \vec{B}) = -\vec{B} \cdot (\vec{A} \times \vec{C})$.

From the result for $L$ we have

$$\frac{d}{dt} \left( \frac{\delta L}{\delta \dot{\vec{R}}} \right) = m \frac{d}{dt} (\dot{\vec{R}} + \vec{\Omega} \times \vec{R})$$
To obtain $\delta L / \delta R$ we rewrite the relevant terms of $L$ using the cyclic property of the triple product as

$$m\hat{R} \cdot (\hat{R} \times \hat{\Omega}) + \frac{m}{2} \left( \hat{\Omega} \times \hat{R} \right) \cdot \left( \hat{\Omega} \times \hat{R} \right)$$

Now the derivative of the first term with respect to $\hat{R}$ is straightforward and for the second term we get

$$\frac{m}{2} \frac{d}{d\hat{R}} \left( \left( \hat{\Omega} \times \hat{R} \right) \cdot \left( \hat{\Omega} \times \hat{R} \right) \right) = m \left( \frac{d}{d\hat{R}} \left( \hat{\Omega} \times \hat{R} \right) \right) \cdot \left( \hat{\Omega} \times \hat{R} \right)$$

or, using the triple product rule

$$= m \left( \frac{d}{d\hat{R}} \left( \hat{R} \right) \right) \cdot \left( \left( \hat{\Omega} \times \hat{R} \right) \times \hat{\Omega} \right)$$

Moving the second term in $\frac{d}{dt} \left( \delta L / \delta \dot{\vec{R}} \right)$ to the right hand side of the Euler-Lagrange equation and adding to the other terms gives the desired result.

(2) Now we use the result of problem 1 to analyse the Foucault pendulum: a mass $m$ suspended by a rigid massless rod of length $L$ projecting normal to the earth surface. The potential $V$ is then the gravitational potential of the earth.

Write $\vec{R} = \vec{R}_E + \vec{r}(t)$ with $\vec{R}_E$ the vector from the center of the earth to the suspension point of the pendulum and $\vec{r}$ giving the position of the mass relative to the suspension point. Note that in the reference frame rotating with the earth, $\vec{R}_E$ is time independent.

(a) Writing

$$V(\vec{R}) = -\frac{mg\vec{R}_E^2}{|\vec{R}_E + \vec{r}|}$$

and expanding for small $\vec{r}$, show that

$$V(\vec{R}) = mg\vec{R}_E \cdot \vec{r} + ...$$

where $g$ is the usual acceleration of gravity at the earth’s surface and the ... denoting terms of order $mgr^2/R_E$ and $mgL/R_E$.

We have

$$|\vec{R}_E + \vec{r}| = \sqrt{\vec{R}_E^2 + \vec{r}^2 + 2\vec{R}_E \cdot \vec{r}} = \vec{R}_E + \vec{R}_E \cdot \vec{r} + ...$$

so

$$\frac{1}{|\vec{R}_E + \vec{r}|} = \left( \vec{R}_E + \vec{R}_E \cdot \vec{r} + ... \right)^{-1} = \frac{1}{\vec{R}_E} - \frac{\vec{R}_E \cdot \vec{r}}{\vec{R}_E^2} + ...$$

Substituting into the equation for $V$ gives the desired result.

Further show that by neglecting terms of order $L/R_E$ and dropping constants and total derivatives the Lagrangian found in problem (1) may be approximated as

$$\mathcal{L} = \frac{m}{2} \dot{\vec{r}}^2 + m\dot{\vec{r}} \cdot \left( \hat{\Omega} \times \dot{\vec{r}} \right) + m\vec{r} \cdot \vec{g}_{eff} + \frac{m}{2} \left( \hat{\Omega} \times \vec{r} \right)^2$$
with 
\[ \vec{g}_{eff} = mg\vec{R}_E - \left( \vec{\Omega} \times \vec{R}_E \right) \times \vec{\Omega} \]

Be sure to explain why the term \( m\vec{r} \cdot \left( \vec{\Omega} \times \vec{R}_E \right) \) does not contribute to the equation of motion and therefore does not have to be written. Suppose you retained it: what would the contribution to the Euler-Lagrange equations be?

From the form for \( L \) given above we have (noting that \( \dot{\vec{R}}_E = 0 \)) we have
\[ L = \frac{m}{2} \left( \dot{\vec{r}} + \vec{\Omega} \times \vec{R}_E + \vec{\Omega} \times \vec{r} \right)^2 - mg\vec{R}_E \cdot \vec{r} \]

Now expanding the square and dropping the constant term \( \left( \vec{\Omega} \times \vec{R}_E \right)^2 \) we get
\[ L = \frac{m}{2} \dot{\vec{r}}^2 + m\vec{r} \cdot \left( \vec{\Omega} \times \vec{R}_E \right) + m\vec{r} \cdot \left( \vec{\Omega} \times \vec{r} \right) + m \left( \vec{\Omega} \times \vec{R}_E \right) \cdot \left( \vec{\Omega} \times \vec{r} \right) + \frac{m}{2} \left( \vec{\Omega} \times \vec{r} \right)^2 - mg\vec{R}_E \cdot \vec{r} \]

Now the term \( m\vec{r} \cdot \left( \vec{\Omega} \times \vec{R}_E \right) \) is a total derivative \( = m \frac{d}{dt} \left( \vec{r} \cdot \left( \vec{\Omega} \times \vec{R}_E \right) \right) \) so makes no contribution to \( \int dt L \); also it would contribute to the Euler-Lagrange equations a term
\[ \frac{d}{dt} \left( \vec{\Omega} \times \vec{R}_E \right) = 0 \]

so it can be dropped.

Using the triple product rule in the next term gives
\[ L = \frac{m}{2} \dot{\vec{r}}^2 + m\vec{r} \cdot \left( \vec{\Omega} \times \vec{R}_E \right) + m\vec{r} \cdot \left( \left( \vec{\Omega} \times \vec{R}_E \right) \times \vec{\Omega} \right) - mg\vec{R}_E \cdot \vec{r} + \frac{m}{2} \left( \vec{\Omega} \times \vec{r} \right)^2 \]

which is the desired result.

(b) Now choose \( x, y, z \) coordinates
\[ \vec{r} = (x, y, z) \]

such that \( z \) is parallel to \( \vec{g}_{eff} \), \( \vec{\Omega} \) is in the \( x - z \) plane
\[ \vec{\Omega} = (\Omega_x, 0, \Omega_z) \]

and \( y \) is the perpendicular direction tangent to the earth’s surface.

Introduce angles \( \theta \) and \( \phi \) so that
\[ x = L\sin\theta\cos\phi \quad y = L\sin\theta\sin\phi \quad z = -L\cos\theta \]
and make the small \( \theta \) approximation
\[ x = L\theta\cos\phi \quad y = L\theta\sin\phi \quad z = -L \left( 1 - \frac{\theta^2}{2} \right) \]

In terms of these coordinates, show that if the terms of order \( \Omega^2 r^2 \) are dropped the Lagrangian can be rewritten as
\[ L = \frac{mL^2}{2} \left( \dot{\theta}^2 + \theta^2 \left( \dot{\phi} - \Omega_z \right)^2 \right) - \frac{mL\bar{g}}{2} \theta^2 \]

with \[ \bar{g} = |\vec{g}_{eff}| - L\Omega_z^2 \]

**Hint:** it is helpful to write the \( \vec{r} \cdot (\vec{\Omega} \times \vec{r}) \) term as \( \vec{\Omega} \cdot (\vec{r} \times \dot{\vec{r}}) \). Because \( \vec{\Omega} \) has only \( x \) and \( z \) components you only need the corresponding components of the cross product. In the small angle approximation the \( x \) component of \( \dot{\vec{r}} \times \vec{r} \) is a total derivative and may be dropped.

The small angle forms of \( x,y,z \) give (keeping only terms of linear order in \( \theta \))

\[
\begin{align*}
\dot{x} &= L\dot{\theta}\cos\phi - L\theta\sin\phi \dot{\phi} \\
\dot{y} &= L\dot{\theta}\sin\phi + L\theta\cos\phi \dot{\phi} \\
\dot{z} &= 0
\end{align*}
\]

so

\[ x^2 + y^2 + z^2 = L^2\theta^2 + L^2\theta^2 \dot{\phi}^2 \]

Now following the hint we write the \( \dot{\vec{r}} \cdot (\vec{\Omega} \times \vec{r}) \) term as \( \vec{\Omega} \cdot (\vec{r} \times \dot{\vec{r}}) \).

The term in \( \vec{r} \times \dot{\vec{r}} \) in the direction of \( x \) is (using \( z = -L + \ldots \))

\[ r_z \dot{r}_y = -L^2\dot{\theta}\sin\phi + L\theta\cos\phi \dot{\phi} \]

which as noted in the hint is a total derivative and will not contribute to the E-L equations.

The term in \( \vec{r} \times \dot{\vec{r}} \) in the direction of \( z \) is

\[ xy - y\dot{x} = L^2\theta^2 \dot{\phi} \]

Thus this contributes a term \( m\Omega_z L^2 \theta^2 \dot{\phi} \) which is the desired cross term. Finally, adding and subtracting the \( \Omega_z^2 \) term gives the desired result.

(c) Write the Euler-Lagrange equations show that one solution is \( \phi(t) = \Omega_z t \) and \( \theta(t) = A\sin\bar{\omega}t \) and give an expression for \( \bar{\omega} \). **NOTE!!** We have dropped terms of order \( \Omega^2 r^2 \) so in your expression for \( \bar{\omega} \) you should not include terms of order \( \Omega^2 L^2 \)

We have

\[ mL^2 \ddot{\theta} = mL^2 (\dot{\phi} - \Omega_z) - mL\bar{g}\theta \]

\[ mL^2 \frac{d}{dt} (\theta^2 (\dot{\phi} - \Omega_z)) = 0 \]

Thus setting \( \phi = \Omega_z t \) makes the second equation automatically satisfied while the first equation becomes

\[ mL^2 \ddot{\theta} = -mL\bar{g}\theta \]

which is the harmonic oscillator equation with characteristic frequency \( \bar{\omega} = \sqrt{\frac{g}{L}} \approx \sqrt{g/L} \).
(d) **In this part you are asked to retain terms of order** $\Omega^2 L^2$.

Specialize to the case where the pendulum is suspended above the earth’s north pole ($\Omega_x = 0$). Show that the $\frac{m}{2} \left( \vec{\Omega} \times \vec{r} \right)^2$ term in $L$ becomes

$$\frac{m}{2} \left( \vec{\Omega} \times \vec{r} \right)^2 \rightarrow \frac{m}{2} \Omega^2 L^2 \theta^2$$

so that the final equation $L$ given in (c) we have

$$\bar{g} = g$$

(in other words the restoring force is exactly the usual gravitational constant).

Comment on this result in light of the solution of the pendulum in the non-rotating reference frame.

Why does this result only hold when the pendulum is suspended above the north (or south) poles

*Returning to our result for $L$ and noting that in this special case $\vec{\Omega}$ and $\vec{R}_E$ are parallel so $\vec{\Omega} \times \vec{R}_E = 0$ we have*

$$L = \frac{m}{2} \dot{r}^2 + m \dot{r} \cdot \left( \vec{\Omega} \times \vec{r} \right) + \frac{m}{2} \left( \vec{\Omega} \times \vec{r} \right)^2 - mg \vec{R}_E \cdot \vec{r}$$

Now $\vec{\Omega} \times \vec{r} = \Omega (-x, y, 0)$ so taking the square gives $\Omega^2 (x^2 + y^2)$ so following the steps in (c) we have

$$L = \frac{mL^2}{2} \left( \dot{\theta}^2 + \dot{\theta}^2 \dot{\phi}^2 - 2\Omega \dot{\theta}^2 \dot{\phi} + \Omega^2 \theta^2 \right) + mgL \theta^2$$

or

$$L = \frac{mL^2}{2} \left( \dot{\theta}^2 + \theta^2 \left( \dot{\phi} - \Omega \right)^2 \right) + mgL \theta^2$$

The arguments of part (c) immediately give us a solution with $\theta$ oscillating with frequency $\omega = \sqrt{g/L}$ and the plane of rotation following the earth’s rotation. This is what you would see if you viewed the pendulum from a frame of reference fixed above the north pole. If the pendulum is not at a pole then, when viewed from a reference frame fixed in space, the position of the pendulum is changing, so we cannot expect the oscillation to be exactly what it would be if the pendulum were suspended in a non-rotating frame.

(3) 10.36

Note that the problem does not specify the point through which the axes of rotation pass. We will choose this to be the origin of coordinates as given in the text.
\[
\begin{align*}
I_{xx} &= ma^2 (0 + 5 + 5) = 10ma^2 \\
I_{yy} &= ma^2 (1 + 4 + 1) = 6ma^2 \\
I_{zz} &= ma^2 (1 + 4 + 1) = 6ma^2 \\
I_{xy} &= -ma^2 (0 + 0 + 0) \\
I_{xz} &= -ma^2 (0 + 0 + 0) \\
I_{yz} &= ma^2 (0 + 2 + 2) = -4ma^2 \\
\end{align*}
\]

or

\[
I = ma^2 \begin{pmatrix}
10 & 0 & 0 \\
0 & 6 & -4 \\
0 & -4 & 6
\end{pmatrix}
\]

The principal axes are

\[
\begin{align*}
e_1 &= \hat{x} \\
e_2 &= \frac{1}{\sqrt{2}} (\hat{y} + \hat{z}) \\
e_3 &= \frac{1}{\sqrt{2}} (\hat{y} - \hat{z})
\end{align*}
\]

and in this basis the matrix is

\[
I = ma^2 \begin{pmatrix}
10 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 10
\end{pmatrix}
\]

\(4\) 10.40

Note: in the absence of torque, \(\vec{L}\) is constant in the space frame, but in the rotating frame the direction of the angular momentum changes. The magnitude cannot change, otherwise \(L\) would change in the space frame.

We have

\[
|L|^2 = L_x^2 + L_y^2 + L_z^2 = \lambda_1 \omega_1^2 + \lambda_2 \omega_2^2 + \lambda_3 \omega_3^2
\]

so

\[
\vec{L} \cdot \frac{d\vec{L}}{dt} = \lambda_1 \omega_1 \omega_1 + \lambda_2 \omega_2 \omega_2 + \lambda_3 \omega_3 \omega_3
\]

The free-body Euler equations are:

\[
\begin{align*}
\lambda_1 \ddot{\omega}_1 &= (\lambda_2 - \lambda_3) \omega_2 \omega_3 \\
\lambda_2 \ddot{\omega}_2 &= (\lambda_3 - \lambda_1) \omega_3 \omega_1 \\
\lambda_3 \ddot{\omega}_3 &= (\lambda_1 - \lambda_2) \omega_1 \omega_2
\end{align*}
\]
so multiplying the first by $\lambda_1 \omega_1$, the second by $\lambda_2 \omega_2$ and the third by $\lambda_3 \omega_3$ and adding gives

$$\vec{L} \cdot \frac{d\vec{L}}{dt} = (\lambda_1 (\lambda_2 - \lambda_3) + \lambda_2 (\lambda_3 - \lambda_1) + \lambda_3 (\lambda_1 - \lambda_2)) \omega_1 \omega_2 \omega_3 = 0$$

(b) The time derivative of the kinetic energy, similarly, is

$$\dot{T} = \lambda_1 \omega_1 \dot{\omega}_1 + \lambda_2 \omega_2 \dot{\omega}_2 + \lambda_3 \omega_3 \dot{\omega}_3$$

so we do the same thing only multiplying the equations by $\dot{\omega}$ not $\lambda \dot{\omega}$ thus getting

$$\dot{T} = ((\lambda_2 - \lambda_3) + (\lambda_3 - \lambda_1) + (\lambda_1 - \lambda_2)) \omega_1 \omega_2 \omega_3 = 0$$

(5) 10.28

The total mass $M$ is related to the mass density $\rho$, the triangle side $2a$ and the height $h$ by

$$M = \rho \sqrt{3} a^2 h$$

The triangular symmetry implies that the off-diagonal components $I_{xz, yz, xy} = 0$. Choose coordinates so that the three vertices of the triangle in the $x$-$y$ plane are $(-a, 0), (0, \sqrt{3}a)$ and $(a, 0)$. The center of mass is then at $x = 0$, $z = h/2$ (by symmetry) and

$$\begin{align*}
y &= \frac{\int_{-a}^{0} dx \int_{0}^{\sqrt{3}(a+x)} y dy + \int_{a}^{0} dx \int_{0}^{\sqrt{3}(a-x)} y dy}{\int_{-a}^{0} dx \int_{0}^{\sqrt{3}(a+x)} dy + \int_{a}^{0} dx \int_{0}^{\sqrt{3}(a-x)} dy} = \frac{a}{\sqrt{3}} \\
I_{zz} &= \frac{M}{\sqrt{3}a^2} \int_{-a}^{0} dx \int_{0}^{\sqrt{3}(a+x)} \left( x^2 + \left( y - \frac{a}{3} \right)^2 \right) dy \\
&\quad + \int_{a}^{0} dx \int_{0}^{\sqrt{3}(a-x)} \left( x^2 + \left( y - \frac{a}{3} \right)^2 \right) dy \\
&= \frac{Ma^2}{3}
\end{align*}$$

while

$$\begin{align*}
I_{xx} = I_{yy} &= \frac{M}{\sqrt{3}a^2 h} \int_{0}^{h} dz \int_{-a}^{0} dx \int_{0}^{\sqrt{3}(a+x)} dy \left( x^2 + \left( z - \frac{h}{2} \right)^2 \right) \\
&\quad + \frac{M}{\sqrt{3}a^2 h} \int_{0}^{h} dz \int_{0}^{a} dx \int_{0}^{\sqrt{3}(a-x)} dy \left( x^2 + \left( z - \frac{h}{2} \right)^2 \right) \\
&= \frac{Mh^2 + a^2}{6}
\end{align*}$$
(6) 10.18

(a) Change in angular momentum about pivot is the impulsive torque about the pivot, in other words

\[ \Delta L = \Delta T = \xi b \]

This implies the angular velocity is \( \omega = \Delta L / I = \xi b / I \) which means that the velocity of the center of mass is \( V_{\text{com}} = a \omega \) so the center of mass momentum \( P_{\text{COM}} = Ma \xi b / I \)

(b) Now, the total angular momentum is the sum of the angular momentum obtained by treating the system as a point particle rotating about the center of mass plus the angular momentum of rotation about the center of mass. Thus the angular momentum of rotation about the center of mass is

\[ L_{\text{rot}} = \xi b - Ma^2 \xi b \]

Because the system is fixed at the pivot point, it is not actually rotating about the center of mass immediately after the impulse, thus the pivot must deliver the impulse needed to cancel \( L_{\text{rot}} \), i.e. must deliver the impulsive torque

\[ T_{\text{pivot}} = -L_{\text{rot}} = \xi b \left( \frac{Ma^2}{I} - 1 \right) \]

To get the force, divide by the distance to the COM, \( a \).

(c) The sweet spot is thus

\[ b_0 = \sqrt{\frac{I}{M}} \]