PHY V1900 Assignment 1

In this problem we will look at the expansion of the universe in a simple non-relativistic, Newtonian limit. What we will do is adapted from Appendix F of S. Weinberg, Cosmology, Oxford University Press (2008). What you need to know are basic facts about derivatives, about volumes and surface areas of spheres, and about Newtonian gravity. I encourage you to discuss this with your classmates. If you have questions come see me!

**Background:** If the energy density of the universe at some length scale is dominated by cold dark matter we can use the equations of Newtonian mechanics to study the expansion of the universe. Suppose that

(i) The universe is characterized by a constant (spatially uniform) density of matter \( \rho \).

(ii) At some initial time \( t_0 \) the matter density is \( \rho_0 \).

(iii) The universe is expanding (or contracting) homogeneously. This means that two points which are separated by a distance \( a_0 \) at time \( t_0 \) will be separated by a distance \( a(t) \) at time \( t \).

(iv) Matter is conserved (neither created nor destroyed).

(i-iv) mean that at time \( t \) the matter density \( \rho(t) \) is

\[
\rho(t) = \rho_0 \left( \frac{a_0}{a(t)} \right)^3
\]

**The questions:**

**A** Choose an arbitrary point as origin and compute, in terms of \( a_0, a(t), da/dt \) and \( \rho_0 \)

- (i) the total mass \( M \) inside a sphere of radius \( R \) centered on the origin
- (ii) the time derivative \( dM/dt \)

**B** Because mass is conserved, a change in the total mass inside the sphere implies a current \( J \) flowing through the surface of the sphere. The current flows in if mass increases and out if mass decreases. If the expansion is uniform the current flow must be perpendicular to the surface of the sphere. Taking \( J \) to be positive for a current flowing out, we have \( dM/dt + J = 0 \). The current density \( j \) is then the total current divided by the surface area of the sphere. The current density \( j \) is also the product of the mass density and the velocity of mass flow \( v \):

\[
j = \rho v
\]

- Using these equations show that the velocity is of the form

\[
v(t) = H(t)R
\]

and give an expression for \( H \) in terms of \( a_0, a(t) \) and \( \rho_0 \).

You have just derived Hubble’s law (in this limit).
To find out how the universe evolves we will use Newton’s laws to compute the change in \( v \) in terms of the forces on a little blob of matter at distance \( R \) from our origin. There are two forces: gravity and a term arising from hydrodynamics.

- (i) In a spherically symmetric situation the force of gravity acting on a particle of mass \( m \) at a distance \( R \) from the origin is the force you would get if you took all the mass within \( R \) and placed it at the origin; in other words, \( F_{\text{gravity}} = -\frac{GMm}{R^2} \) with \( M \) the total mass within \( R \) and \( G \) the gravitational constant. Please compute the gravitational force on a little region of matter, of unit mass, at distance \( R \).

- (ii) There is a term arising from hydrodynamics which also plays the role of a force. In this problem it is (for a region of matter of unit mass)

\[
F_{\text{hydro}} = -mv \frac{dv}{dR}
\]

Please compute \( F_{\text{hydro}} \).

- (iii) Newton’s law (for a region of fluid of unit mass) is then \( F = m\frac{dv}{dt} \). Please use this and your previous results to get an equation for the time dependence of \( a \) (I am not asking you to solve this equation).

The equation you found in (Ciii) implies

\[
\frac{1}{2} \left( \frac{da}{dt} \right)^2 - \frac{A}{a(t)} = E
\]

with \( A \) and \( E \) constants independent of time. (Check this by differentiating both sides of the equation with respect to time and using your result from part C). If \( E > 0 \) then one can have a solution with \( a \to \infty \) and \( da/dt = \sqrt{2E} \) (universe expands forever) ; \( E < 0 \) implies that there is a largest value of \( a = -A/E \) (universe is trapped in its potential well).

- In the marginal case \( E = 0 \) show that one can have a uniformly decelerating but never ceasing expansion \( a(t) = C_0t^{2/3} \) and find \( C_0 \).

Power laws in marginal cases are common.