Matrices

Matrices are a way of representing systems of linear equations. Consider the following systems of equations relating variables $s, t$ to $u, v$:

\[
\begin{align*}
s &= au + bv \\
t &= cu + dv
\end{align*}
\]  

(1) (2)

This system of equations can be represented by the following matrix equation.

\[
\begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}
\]  

(3)

Why would we want to do this? It turns out to make calculations easier. Imagine we now want use new variables $p, q$, which are related to $s, t$:

\[
\begin{align*}
p &= es + ft \\
q &= gs + ht
\end{align*}
\]  

(4) (5)

Solving for $p$ and $q$ in terms of the original variables $s$ and $t$ quickly becomes a huge mess, for example:

\[
p = e(au + bv) + f(cu + dv); q = g(au + bv) + h(cu + dv)
\]  

(6)

This really isn’t so bad, but if there were more than two variables involved you can see how quickly this becomes a mess. In terms of matrices we would have:

\[
\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}
\]  

(7)

We can multiply the matrices together to have:

\[
\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} ea + fc & eb + fd \\ ga + hc & gb + hd \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}
\]  

(8)

The matrix equation can also become messy, but it is nice because you can figure out the composition rules without explicitly using the parameters $s, t$. In many cases the matrices will have additional properties that simplify the algebra.

So in summary: matrices aren’t new, they are just a new way of representing systems of linear equations. The multiplication rules for $2 \times 2$ matrices, and a $2 \times 2$ matrix times a vector (which can be thought of as a $2 \times 1$ matrix) are given above. You should also know the identity matrix:

\[
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]  

(9)

Any matrix or vector is unchanged when multiplied by the identity, try it! Also, in general matrix multiplication is *not commutative*, that as $AB \neq BA$ if $A$ and $B$ are matrices. You should also convince yourself this is true.