

1. THE MUON MAGNETIC MOMENT

A charged particle with mass $m$ and charge $q$ has a magnetic moment that arises from its spin $\vec{S}$:

$$\vec{\mu} = g_s \frac{q}{2m} \vec{S}$$ (1)

For a spin-$\frac{1}{2}$ particle, $\vec{S} = \frac{\hbar}{2} \vec{\sigma}$, and $g_s$ is called the Lande g-factor. From classical calculations that treat the charged particle as a rotating piece of current, $g_s = 1$. Surprisingly, the Dirac equation in relativistic quantum mechanics predicts that $g_s = 2$, a result which agreed with the crude experiments of the time. In the 1940’s, a new understanding of renormalization in the developing quantum field theory of electrodynamics allowed physicists like Feynman and Schwinger to make an unambiguous prediction that $g_s > 2$ for the electron.

This magnetic moment is noticeable when the particle is placed in an external field, because the Hamiltonian gains a term

$$H = -\vec{\mu} \cdot \vec{B}.$$ (2)

The spin will precess around the direction of the field with some frequency, and this is a way to measure the magnetic moment. Typically, we refer to the anomalous magnetic moment $a_{\mu}$ instead of the full g-factor. For the muon:

$$a_{\mu} = \frac{1}{2}(g_{\mu} - 2)$$ (2)

This value is a “clean” observable: the theory is well enough understood to calculate it unambiguously and the experimental situation is such that it can be measured to high precision. The theoretical and experimental determination of $a_{\mu}$ is similar to that for $a_{e}$, but they differ in the details.

2. THEORETICAL CALCULATION

In essence, we are interested in all of the ways that the muon can interact with the magnetic field. Using Feynman diagrams, this corresponds to any diagram with a muon coming in, a muon coming out, and an external photon line. This is the most general $\text{E&M}$ interaction.

The full interaction is pictured symbolically in Figure 1, and the cross section can estimated to increasing order using perturbation theory. The Dirac prediction of $g = 2$ corresponds to the tree-level diagram.

Since the bulk of the results will come from $\text{QED}$ corrections, it makes sense to expand the full moment in powers of $\alpha$, the electromagnetic coupling constant. Since each additional pair of vertices will add another power of $\alpha$, it makes sense to look at every diagram of increasing orders. In particular,

$$a_{\mu}^{\text{full}} = \sum_{n=1}^{\infty} a_{\mu}^{(2n)} \left(\frac{\alpha}{\pi}\right)^n$$ (3)

2.1. QED contribution

The lion’s share of the anomalous magnetic moment comes from contributions to the cross-section from higher-order diagrams involving only photons and leptons. Let’s structure these contributions in a way that illuminates which ones exist for all leptons, and where the muon specifically comes into play.

2.1.1. Same-flavor loops

Let’s restrict the diagrams further and consider only Feynman diagrams with photons and the same flavor lep-
FIG. 3: Examples of two-loop QED corrections. All internal fermions have the same flavor as the external lines.

There is a single one-loop correction, shown in Figure 2. This is the celebrated Schwinger “radiative correction” derived in the late 1940’s, with the contribution given by

$$a_1^2 = \frac{1}{2} \left( \frac{\alpha}{\pi} \right)$$

(4)

There are seven two-loops diagrams, pictured in Figure 3. It turns out that all of these are exactly integrable. The answer can be stated in closed form, although this involves the zeta function $\zeta(x)$. The numerical value is roughly:

$$a_4^2 = -0.328 \ldots \left( \frac{\alpha}{\pi} \right)^2$$

(5)

There are seventy-two three-loop diagrams, some examples of which are given in Figure 4. Remarkably, all of these contributions are also known analytically, though the closed form now involves polylogarithmic functions. The calculations at this stage are getting much harder. The numerical value is:

$$a_6^2 = 1.181 \ldots \left( \frac{\alpha}{\pi} \right)^3$$

(6)

At the four-loop level, there are 891 diagrams. Some of these are solved exactly, but others require numerical calculation techniques. This value is continuously improved as theorists develop new methods and computing power increases. The present contribution is only known to within an uncertainty:

$$a_8^4 = -1.7283(35) \ldots \left( \frac{\alpha}{\pi} \right)^4$$

(7)

It turns out that the signs alternate and the coefficients remain fairly small. Theorists are not sure why, a priori, either of these things is true.

The five-loop case contains over twelve thousand diagrams. Since $\left( \frac{\alpha}{\pi} \right)^5 \approx 7 \times 10^{-14}$, current experiments are insensitive to corrections of this order.

2.1.2. Vacuum polarization with $e^\pm$ and $\tau^\pm$

Up to this point, we have assumed that all internal fermion loops have the same flavor as the external lines. In reality, any of the internal loops in Figures 3 and 4 could have been $e^+ e^-$ or $\tau^+ \tau^-$ pairs (in the case of the muon). This is an example of vacuum polarization. That is, any photon propagator is the sum of the amplitudes of propagators which have any number of internal loops.

We must reconsider all of the Feynman diagrams with internal fermion loops and add a contribution that includes electron and tau loops. The amplitude of these different-flavor diagrams varies with the ratios...

FIG. 4: Examples of three-loop QED corrections. Again, internal fermions have the same flavor as the external lines.
At the two-loop level, there is only one diagram to be reconsidered, and we can call these contributions $a_{\mu,vp}^4(m_{\mu})$ and $a_{\mu,vp}^4(m_{e})$. At the three-loop level, the mixed contribution $a_{\mu,vp}^6(m_{\mu}, m_{e}, m_{\tau})$ must also be considered, and so on.

The contributions from $\tau^{\pm}$ internal loops drop out quickly, and this correction is dominated by diagrams with $e^{\pm}$ internal loops.

2.1.3. Light-by-light scattering

In addition, there is a contribution from an interesting process called light by light scattering, an example of which is shown in Figure 5. A small discrepancy between experiment and theory in the 1970’s actually led theorists to reconsider the importance of this diagram, to which current experiments are sensitive. The corrections start at the $a_{\mu}^6$ level.

$$m_{\mu} \approx \frac{105 \text{ MeV}}{511 \text{ keV}} \approx 205$$ \hspace{1cm} (8)

$$m_{\mu} \approx \frac{105 \text{ MeV}}{1777 \text{ MeV}} \approx 0.06$$ \hspace{1cm} (9)

2.1.4. Total

Through the $a_{\mu}^6$ level, the current theoretical contribution from all purely-QED processes is

$$a_{\mu}^{QED} = 116584714.12(0.39)(0.04) \times 10^{-11}$$ \hspace{1cm} (10)

There are contributions from the weak interaction. There are many possibilities involving the $W^{\pm}$, $Z^0$ and even the Higgs boson. The original diagram in Figure 1 can proceed, for example, by any of the one-loop corrections in Figure 6. Since $m_{W} \approx 80.4$ GeV and $m_{Z} \approx 91.2$ GeV, these particles are highly virtual and the interactions are suppressed. Thus, even the leading-order contributions are very small. The total theoretical contribution, with uncertainties dominated by the Higgs mass and second-order effects, is

$$a_{\mu}^{weak} = 154(4)(2) \times 10^{-11}$$ \hspace{1cm} (11)

2.2. Weak contribution

Similarly, vacuum polarization can occur with hadrons such as $q\bar{q}$ pairs in the internal loops. These contributions start at the two-loop level. Mixed hadron/lepton polarization contributions at the three-loop level, and there is even a similar light-by-light hadronic scattering contribution similar to Figure 5. There are numerous theoretical uncertainties in how to calculate these higher-order diagrams, but the resulting contributions are small. In particular, since internal lines with a particle of mass $M$ contribute a factor of $(m_{M})$ (and $m_{\mu} \approx 205 m_{e}$), this channel is thought to be a way of detecting new physics that is not as accessible in electron $(g-2)$ measurements.
2.4. Total theoretical prediction

Summing the QED, weak and hadronic contributions, the current value is

\[ a^\text{th}_\mu = 116591785(61) \times 10^{-11} \]  

(12)

3. EXPERIMENTAL DETERMINATION

Just as in the theoretical determination, we need to consider the most general way in which the muon interacts with an external magnetic field. The precession frequency of the spin around the direction of the \( \vec{B} \)-field is

\[ w_s = -g_s \frac{qB}{2m} - \frac{qB}{m} \left( \frac{1 - \gamma}{\gamma} \right) \]  

(13)

This is the well-known Larmor result. This was the quantity measured in the original muon \( (g - 2) \) experiment at Nevis. The cyclotron frequency, that is, the frequency of rotation of \( \vec{p} \) is

\[ w_c = \frac{qB}{m\gamma} \]  

(14)

We can form another quantity \( w_a = w_s - w_c \), which is the precession of the spin relative to the momentum. This is called the anomalous frequency difference, which for the simple forms of Equations 13 and 14 is

\[ w_a = -a_\mu \frac{qB}{m} \]  

(15)

If we knew the initial and final polarization of muons in flight, then measuring the frequency gives us a measurement of the anomalous moment. Importantly, the difference in the frequencies does not depend on the momentum of the muon, a fact we will exploit later. This was the quantity measured in the first iteration of the CERN experiments.

3.1. Experimental setup

The source of muons is obtained from pion decay by running a proton beam through a spallation target. Because of dynamic considerations, the charged pion decays preferentially (the branching ratios are close to 100%) to a muon and a neutrino, with the charged lepton carrying away most of the energy.

\[ \pi^+ \to \mu^+ + \nu_\mu \]  

(16)

\[ \pi^- \to \mu^- + \bar{\nu}_\mu \]  

(17)

The neutrino escapes, but the muon is then injected into a cyclotron. Since the pion has spin zero, conservation of angular momentum and the left-handed character of the weak interaction sets the helicity of the escaping muon as it enters the magnetic field. We can select the appropriate type of muon by selecting the right charge.

The muon decays in-flight to an electron and two neutrinos. In the rest frame, the latter two are typically emitted in the direction opposite of the \( e^\pm \).

\[ \mu^- \to \nu_e + \bar{\nu}_\mu + e^- \]  

(18)

\[ \mu^+ \to \bar{\nu}_e + \nu_\mu + e^+ \]  

(19)

Helicity arguments show that the electron comes out with the same polarity as the parent muon. Since the electron has a much smaller mass, its radius of curvature in the same \( \vec{B} \)-field is much smaller, so it spirals inward and hits a calorimeter inside the cyclotron. Thus, we know the initial and final polarization of the muon.

3.2. Magic \( \gamma \)

In further iterations of the CERN experiments, an electrostatic quadrupole field was introduced in the cyclotron to focus the beam. With this new innovation, the equations of motion became more complicated. Generally, the E and B fields seen by the relativistic muon are a function of \( \vec{\beta} \cdot \vec{E}, \vec{\beta} \cdot \vec{B}, \vec{\beta} \times \vec{E} \) and \( \vec{\beta} \times \vec{B} \). The anomalous frequency difference takes the form

\[ w_a = \frac{q}{m} \left( a_\mu B - a_\mu \frac{\gamma}{1 - \gamma} \left( \vec{\beta} \cdot \vec{B} \right) \vec{\beta} \right) \] 

\[ - \left( a_\mu - \frac{1}{\gamma^2 - 1} \right) \left| \frac{\vec{\beta} \times \vec{E}}{c} \right| \]  

(20)

In a cyclotron, the momentum lies within the horizontal plane and the magnetic field is vertical, so the second term is proportional to \( \vec{\beta} \cdot \vec{B} = 0 \).

The third time is undesirable because it is a complicated function of the electric field. Since we have not yet constrained the momentum of the particle, we can set \( \gamma = \sqrt{1/a_\mu + 1} \approx 29.3 \). This is known as the “magic \( \gamma \)”, for which the equations of motion take on a particularly simple form. Only the first term survives:

\[ w_a = a_\mu \frac{q}{mB} \]  

(21)

This Lorentz factor, which requires selecting pions of momentum \( p \approx 3.1 \text{ GeV}/c \), is actually rather favorable. At this value of \( \gamma \), the radius of curvature is around 14m, and the time-dilated muons (\( \beta \approx 0.9994 \)) live long enough to sample the track several dozen times - enough that
averaging over the inhomogeneities in the magnetic field is a good approximation of the field seen by the particle. As in other areas of physics, this frequency-dependent measurement is one of the more precise techniques that exists to measure fundamental values. In reality, the other terms in Equation 20 are non-zero, caused by a slightly mis-aligned field and edge effects \((\vec{\beta} \cdot \vec{B} \neq 0)\) and electrons slightly off the magic \(\gamma (\vec{a}_\mu (\vec{B} \times \vec{E}))\). In the experiments that measure \((g-2)\), physicists account for these imperfections.

4. MEASUREMENT HISTORY

Though the anomalous moment was predicted as early as 1947 by Schwinger et. al., it was confirmed only in 1961 by experiments at Columbia University’s Nevis Laboratories. Additionally, this experiment was important because it showed that the muon functioned like a heavier electron, giving insight into the nature of charged leptons. It confirmed that the muon was a structureless point particle with no short-distance effects. These experiments were done with \(\mu^+\) only, and confirmed the theoretical prediction of the anomaly to the \((\frac{g}{2})^2\) level.

The small \(a_\mu\) excess was measured to be

\[
a_\mu = 122(8) \times 10^{-5} \tag{22}
\]

The next generation of experiments took place at CERN, which started in the early 60’s and ran until 1979. The three generations of experiments, called CERN I, II and III, featured substantial improvements in technique and precision. CERN I improved on the Nevis result, measuring to the \((\frac{g}{2})^2\) level. CERN II was the first experiment to use both \(\mu^\pm\) and had sensitivity to the \((\frac{g}{2})^3\) level. This is important because confirming that \(a_\mu\) is the same for muons and antineutrons serves as a high-precision test of the CPT Theorem.

After CERN II measured the anomalous moment to a precision of 256ppm, a 1.7\(\sigma\) disparity between experiment and the current theoretical value was found. This led to a re-evaluation of the QED contributions, specifically the light-by-light scattering process above. CERN III introduced the electrostatic focusing technique that required fixing the magic \(\gamma\) but also dramatically improved experimental results. By 1979, the experiment was sensitive to QED effects to \((\frac{g}{2})^3\) order, plus hadronic effects. With an accuracy of 7.3ppm, the value was determined to be

\[
a_\mu = 165911(11) \times 10^{-8} \tag{23}
\]

The E821 Collaboration used the key electrostatic focusing technique, and made numerous improvements to the CERN III experimental setup. The experiment ran until 2001 at Brookhaven National Laboratory and was sensitive to the \((\frac{g}{2})^4\) QED level, as well as hadronic and even weak effects. With an accuracy of 0.54ppm, the experiment had such a good grasp of systematics that two-thirds of the error is due to statistics! The final value is

\[
a_\mu = 116592080(63) \times 10^{-11} \tag{24}
\]

5. SUMMARY

The muon \((g-2)\) is one of the most accurately predicted and measured physical constants. It provides a direct, precise test of QED and tests our knowledge of the weak and strong interactions as well. Not only does the next theoretical prediction lie at the five-loop level, but the experiment is sensitive to extremely sophisticated effects such as hadronic light-by-light scattering. (There is a 7\(\sigma\) excess between experiment and theory if the strong and weak effects are not considered.) In fact, the theoretical value is so precise that it is affected by uncertainties in such fundamental constants as the fine structure constant and the tau mass.

The history of the muon anomalous magnetic moment is one of interplay between experiment and theory. Comparing the best theoretical prediction in (12) and the most recent experimental result in (24), there is a discrepancy of 295(88) \times 10^{-11}. This is a 3.4\(\sigma\) result, and scientists will push the limits of our knowledge of the Standard Model attempting to resolve it.

A more comprehensive treatment can be found in two excellent papers by Jegerlehner[1] and Miller et. al.[2]