$$
\begin{aligned}
&= \int_{\partial H_+} \nu_+ + \int_{\partial H_-} \nu_- = \int_{S^1} \nu_+ + \int_{(-S^1)} \nu_- \\
&= \int_{S^1} d\phi + \int_{(-S^1)} (-d\phi) = 2\pi + 2\pi = 4\pi\,.
\end{aligned} \tag{1.225}
$$

Here, we have split the volume integral over S^2 into the sum over the two hemispheres, and in each case we have replaced the volume-form ω by its expression as the exterior derivative of a 1-form that is globally-defined within that hemisphere. Now, we we apply Stokes' theorem, we convert the volume integrals over hemispheres into integrals around their boundaries (i.e. the equatorial circle). We must be careful about the orientations of the circles; we have that ∂H_+ is the positively-oriented equatorial circle, but ∂H_- has the opposite orientation. Thus, when we put the two contributions together, we correctly recover the 2-dimensional "volume" of the unit S^2.

1.13 The Levi-Civita Tensor and Hodge Dualisation

1.13.1 The Levi-Civita Tensor

The totally-antisymmetric tensor ε_{ijk} in 3-dimensional Cartesian tensor calculus is a familiar object. It is defined by saying that ε_{ijk} is $+1$, -1 or 0 depending on whether ijk is an even permutation of 123, an odd permutation, or no permutation at all (such as 112). We already introduced an analogous n-dimensional totally-antisymmetric object $\varepsilon^{i_1\cdots i_n}$ in equation (1.215). However, we must be careful; this object is *not* a tensor under general coordinate transformations.

Let us first of all define $\varepsilon_{i_1\cdots i_n}$ with downstairs indices. We shall say

$$
\varepsilon_{i_1\cdots i_n} = \pm 1, 0\,, \tag{1.226}
$$

where we have $+1$ if $\{i_1\ \ldots, i_n\}$ is an even permutation of the numerically-ordered index values $\{1,\ldots,n\}$, we have -1 if it is an odd permutation, and we have 0 if it is no permutation at all. We define $\varepsilon_{i_1\cdots i_n}$ to have these values in *all* coordinate frames, which means that, *by definition*, we have

$$
\varepsilon'_{i_1\cdots i_n} = \varepsilon_{i_1\cdots i_n}\,. \tag{1.227}
$$

Is it a tensor? The answer is no, and we can prove this by showing that it does not transform as a tensor. Suppose it did, and so we start in a coordinate frame x^i with the components being ± 1 and 0, as defined above. We could then work out its components in a primed frame, giving

$$
\tilde\varepsilon_{i_1\cdots i_n} = \frac{\partial x^{j_1}}{\partial x'^{i_1}} \cdots \frac{\partial x^{j_n}}{\partial x'^{i_n}}\, \varepsilon_{j_1\cdots j_n}\,. \tag{1.228}
$$

(We avoid using $\varepsilon'_{i_1\cdots i_n}$ to denote the transformed components in the primed frame because we are currently *testing* whether the transformed components, calculated assuming that $\varepsilon_{i_1\cdots i_n}$ is a tensor, agree with our *definition* of $\varepsilon'_{i_1\cdots i_n}$ given in (1.227). As we shall see, they do not agree.)

The right-hand side of (1.228) can be recognised as giving

$$\Big|\frac{\partial x}{\partial x'}\Big|\, \varepsilon_{i_1\cdots i_n}\,, \tag{1.229}$$

where $\left|\frac{\partial x}{\partial x'}\right|$ is the Jacobian of the transformation, i.e. the determinant of the transformation matrix $\partial x^j/\partial x'^i$. This follows from the identity that

$$M^{j_1}{}_{i_1}\,\cdots M^{j_n}{}_{i_n}\varepsilon_{j_1\cdots j_n} = \det(M)\,\varepsilon_{i_1\cdots i_n} \tag{1.230}$$

for any $n\times n$ matrix. (Check it for $n=2$, if you doubt it.) Since (1.229) is not simply equal to $\varepsilon_{i_1\cdots i_n}$, we see that $\varepsilon_{i_1\cdots i_n}$, defined to be ±1 and 0 in *all* frames, does not transform as a tensor. Instead, it is what is called a *Tensor Density.*

A quantity with components $H_{i_1\cdots i_p}$ is said to be a tensor density of weight w if it transforms as

$$H'_{i_1\cdots i_p} = \Big|\frac{\partial x'}{\partial x}\Big|^w\, \frac{\partial x^{j_1}}{\partial x'^{i_1}}\cdots \frac{\partial x^{j_1}}{\partial x'^{i_1}}\, H_{j_1\cdots j_p}\,, \tag{1.231}$$

under general coordinate transformations. Of course ordinary tensors, for which $w=0$, are the special case of tensor densities of weight 0.

Noting that $\left|\frac{\partial x'}{\partial x}\right| = \left|\frac{\partial x}{\partial x'}\right|^{-1}$, we see from (1.229) that $\varepsilon_{i_1\cdots i_n}$ transforms as a tensor density of weight 1 under general coordinate transformations, namely

$$\varepsilon'_{i_1\cdots i_n} = \Big|\frac{\partial x'}{\partial x}\Big|\, \frac{\partial x^{j_1}}{\partial x'^{i_1}}\cdots \frac{\partial x^j_n}{\partial x'^{i_n}}\, \varepsilon_{j_1\cdots j_n}\,. \tag{1.232}$$

Furthermore, it is indeed an *invariant* tensor density, i.e. $\varepsilon'_{i_1\cdots i_n} = \varepsilon_{i_1\cdots i_n}$; it takes the same numerical values in all coordinate frames.

We can make an honest tensor by multiplying $\varepsilon_{i_1\cdots i_n}$ by a scalar density of weight -1. Such an object can be built from the metric tensor. Consider taking the determinant of the inverse metric. Since we have already introduced the notation that $g\equiv\det(g_{ij})$, it follows that we shall have $\det(g^{ij}) = 1/g$. Thus we may write

$$\frac{1}{g} = \frac{1}{n!}\, g^{i_1 j_1}\cdots g^{i_n j_n}\, \varepsilon_{i_1\cdots i_n}\, \varepsilon_{j_1\cdots j_n}\,. \tag{1.233}$$

(Again, if this is not obvious to you, check it for the case $n=2$.) Changing to a primed coordinate system, and recalling that $\varepsilon_{i_1\cdots i_n}$ is an invariant tensor density, we therefore have

$$\frac{1}{g'} \;=\; \frac{1}{n!}\, g'^{i_1 j_1}\cdots g'^{i_n j_n}\, \varepsilon_{i_1\cdots i_n}\, \varepsilon_{j_1\cdots j_n}$$

$$
\begin{aligned}
&= \frac{1}{n!}\, g^{k_1\ell_1}\cdots g^{k_n\ell_n}\,\frac{\partial x'^{i_1}}{\partial x^{k_1}}\cdots\frac{\partial x'^{i_n}}{\partial x^{k_n}}\,\frac{\partial x'^{j_1}}{\partial x^{\ell_1}}\cdots\frac{\partial x'^{j_n}}{\partial x^{\ell_n}}\,\varepsilon_{i_1\cdots i_n}\,\varepsilon_{j_1\cdots j_n}\\
&= \Big|\frac{\partial x'}{\partial x}\Big|^2\,\frac{1}{g}\,.
\end{aligned}
\qquad (1.234)
$$

This shows that $g' = \Big|\frac{\partial x'}{\partial x}\Big|^{-2} g$; i.e. that g is a scalar density of weight -2. Hence $\sqrt{|g|}$ is a scalar density of weight -1, and so we may define the tensor (i.e. with weight 0)

$$
\epsilon_{i_1\cdots i_n} \equiv \sqrt{|g|}\,\varepsilon_{i_1\cdots i_n}\,. \qquad (1.235)
$$

We shall universally use the notation $\epsilon_{i_1\cdots i_n}$ for the honest tensor, and $\varepsilon_{i_1\cdots i_n}$ for the tensor density whose components are $\pm1, 0$. The totally-antisymmetric tensor $\epsilon_{i_1\cdots i_n}$ is called the *Levi-Civita tensor.*

Some further remarks are in order at this point. First, we shall *always* define $\varepsilon_{i_1\cdots i_n}$ to be $+1$ if its indices are an even permutation of the numerically-ordered index values $1,\ldots,n$, to be -1 for an odd permutation, and 0 for no permutation. For the tensor density with upstairs indices, we *define* them to be numerically given by

$$
\varepsilon^{i_1\cdots i_n} \equiv (-1)^t\,\varepsilon_{i_1\cdots i_n}\,, \qquad (1.236)
$$

where t is the number of negative eigenvalues of the metric g_{ij}. The typical cases will be $t=0$ if we are doing Riemannian geometry, and $t=1$ in special or general relativity.

The second remark is to note that $\varepsilon^{i_1\cdots i_n}$ is *not* given by raising the indices on $\varepsilon_{i_1\cdots i_n}$ using inverse metrics. This is the *one and only* exception to the otherwise universal rule that when we use the same symbol on an object with upstairs indices and an object with downstairs indices, the former is related to the latter by raising the indices with inverse metrics.

The third remark is that $\varepsilon^{i_1\cdots i_n}$ is a tensor density of weight -1. Thus we have tensors $\epsilon_{i_1\cdots i_n}$ and $\epsilon^{i_1\cdots i_n}$ related to the corresponding tensor-densities by

$$
\epsilon_{i_1\cdots i_n} = \sqrt{|g|}\,\varepsilon_{i_1\cdots i_n}\,, \qquad \epsilon^{i_1\cdots i_n} = \frac{1}{\sqrt{|g|}}\,\varepsilon^{i_1\cdots i_n}\,. \qquad (1.237)
$$

Note that $\epsilon^{i_1\cdots i_n}$ *is* obtained by raising the indices on $\epsilon_{i_1\cdots i_n}$ with inverse metrics. This accords with our second remark above.

The fourth remark is that if the number of negative eigenvalues t of the metric is odd, then the determinant g is negative. This is why we have written $\sqrt{|g|}$ in the definitions of the totally-antisymmetric tensors $\epsilon_{i_1\cdots i_n}$ and $\epsilon^{i_1\cdots i_n}$. If we know we are in a situation where $t=0$ (or more generally $t=$even), we typically just write $\sqrt{g}$. If on the other hand we know we are in a situation where $t=1$ (or more generally $t=$odd), we typically write $\sqrt{-g}$.

There are some very important identities that are satisfied by the product of two Levi-Civita tensors. Firstly, one can establish that

$$\epsilon^{i_1\cdots i_n}\,\epsilon_{j_1\cdots j_n} = n!\,(-1)^t\,\delta^{i_1\cdots i_n}_{j_1\cdots j_n}\,, \tag{1.238}$$

where as usual t is the number of negative eigenvalues of the metric, and we have defined

$$\delta^{i_1\cdots i_p}_{j_1\cdots j_p} \equiv \delta^{[i_1}_{[j_1}\cdots\delta^{i_n]}_{j_n]}\,. \tag{1.239}$$

Note that for any antisymmetric tensor $A_{i_1\cdots i_p}$ we have

$$A_{i_1\cdots i_p}\,\delta^{i_1\cdots i_p}_{j_1\cdots j_p} = A_{j_1\cdots j_p}\,. \tag{1.240}$$

It is quite easy to prove (1.238) by enumerating the possible sets of choices for the index values on the left-hand side and on the right-hand side, and verifying that the two expressions agree. Of course one need not verify every single possible set of index assignments, since both the left-hand side and the right-hand side are manifestly totally antisymmetric in the i indices, and in the j indices. In fact this means one really only has to check one case, which could be, for example, $\{i_1,\ldots,i_n\}=\{1,\ldots,n\}$ and $\{j_1,\ldots,j_n\}=\{1,\ldots,n\}$. With a little thought, it can be seen that once the two sides are shown to agree for this set of index choices, they *must* agree for any possible set of index choices.

It is also useful to record the expression one gets if one contracts p of the indices on a pair of Levi-Civita tensors. The answer is

$$\epsilon^{i_1\cdots i_q k_1\cdots k_p}\,\epsilon_{j_1\cdots j_q k_1\cdots k_p} = p!\,q!\,(-1)^t\,\delta^{i_1\cdots i_q}_{j_1\cdots j_q}\,, \tag{1.241}$$

where we have defined $q\equiv n-p$ in n dimensions. The proof is again just a matter of enumerating inequivalent special cases, and checking the equality of the two sides of the equation for each such case. Again, if one spends enough time thinking about it, one eventually sees that it is almost trivially obvious. Note that (1.238) is just the special case of (1.241) when $p=0$.

As an example, in three dimensions with positive-definite metric signature, we have

$$\begin{aligned}
\epsilon^{ijk}\,\epsilon_{\ell mn} &= 6\delta^{ijk}_{\ell mn} = \delta^i_\ell\delta^j_m\delta^k_n + \delta^j_\ell\delta^k_m\delta^i_n + \delta^k_\ell\delta^i_m\delta^j_n - \delta^i_\ell\delta^k_m\delta^j_n - \delta^j_\ell\delta^i_m\delta^k_n - \delta^k_\ell\delta^j_m\delta^i_n\,,\\
\epsilon^{ijm}\,\epsilon_{k\ell m} &= 2\delta^{ij}_{k\ell} = \delta^i_k\delta^j_\ell - \delta^j_k\delta^i_\ell\,.
\end{aligned} \tag{1.242}$$

These, or at least the second identity, should be very familiar from Cartesian tensor analysis.

1.13.2 The Hodge dual

Suppose we have a p-form ω in n dimensions. It is easy to count the number N_p of independent components $\omega_{i_1\cdots i_p}$ in a general such p-form: the antisymmetry implies that the answer is

$$N_p = \frac{n!}{p!\,(n-p)!}\,. \tag{1.243}$$

For example, for a 0-form we have $N_0 = 1$, and for a 1-form we have $N_1 = n$. These are exactly what one expects for a scalar and a co-vector. For a 2-form we have $N_2 = \frac{1}{2}n(n-1)$, which again is exactly what one expects for a 2-index antisymmetric tensor (it is just like counting the independent components of a general $n \times n$ antisymmetric matrix).

It will be noticed from (1.243) that we have

$$N_p = N_{n-p}\,, \tag{1.244}$$

i.e. the number of independent components of a p form is the same as the number of independent components of an $(n-p)$-form in n dimensions. This suggests the possibility that there could exist a 1-1 mapping between p-forms and $(n-p)$-forms, and indeed precisely such a mapping exists. It is called *Hodge Duality*, and it is implemented by means of the Levi-Civita tensor.

Suppose a p-form ω is expanded in a coordinate basis in the usual way, as

$$\omega = \frac{1}{p!}\,\omega_{i_1\cdots i_p}\,dx^{i_1}\wedge\cdots\wedge dx^{i_p}\,. \tag{1.245}$$

We can define a *Hodge dual* basis for $q = n-p$ forms, as

$$*(dx^{i_1}\wedge\cdots\wedge dx^{i_p}) = \frac{1}{q!}\,\epsilon_{j_1\cdots j_q}{}^{i_1\cdots i_p}\,dx^{j_1}\wedge\cdots dx^{j_q}\,. \tag{1.246}$$

We can then read off the Hodge dual of ω, namely

$$*\omega = \frac{1}{p!\,q!}\,\epsilon_{j_1\cdots j_q}{}^{i_1\cdots i_p}\,\omega_{i_1\cdots i_p}\,dx^{j_1}\wedge\cdots dx^{j_q}\,. \tag{1.247}$$

Comparing with the standard definition of a q-form, we can therefore read off the components of the q-form $*\omega$, whose expansion is

$$*\omega = \frac{1}{q!}\,(*\omega)_{j_1\cdots j_q}\,dx^{j_1}\wedge\cdots\wedge dx^{j_q}\,. \tag{1.248}$$

Thus from (1.247) we read off

$$(*\omega)_{j_1\cdots j_q} = \frac{1}{p!}\,\epsilon_{j_1\cdots j_q}{}^{i_1\cdots i_p}\,\omega_{i_1\cdots i_p}\,. \tag{1.249}$$

Equation (1.249) gives the mapping from the p-form ω to its Hodge dual, the $q = n - p$ form $*\omega$. It was said earlier that this is a 1-1 mapping, and so we must be able to invert it. This is easily done, by making use of the identity (1.241) for the contraction of two Levi-Civita tensors on some of their indices. Thus, taking the Hodge dual of the Hodge dual of ω, making use of the basic defining equation (1.249), we shall have

$$\begin{aligned} (**\omega)_{i_1\cdots i_p} &= \frac{1}{p!\, q!}\, \epsilon_{i_1\cdots i_p}{}^{j_1\cdots j_q}\, \epsilon_{j_1\cdots j_q}{}^{k_1\cdots k_p}\, \omega_{k_1\cdots k_p} \\ &= \frac{(-1)^{pq}}{p!\, q!}\, \epsilon_{i_1\cdots i_p}{}^{j_1\cdots j_q}\, \epsilon^{k_1\cdots k_p}{}_{j_1\cdots j_q}\, \omega_{k_1\cdots k_p} \\ &= \frac{(-1)^{pq+t}}{p!\, q!}\, p!\, q!\, \delta^{k_1\cdots k_p}_{i_1\cdots i_p}\, \omega_{k_1\cdots k_p} \\ &= (-1)^{pq+t}\, \omega_{i_1\cdots i_p}\,. \end{aligned} \tag{1.250}$$

In getting to the second line, the shifting of the block of q indices $(j_1 \cdots j_q)$ through the block of p indices $(k_1 \cdots k_p)$ on the second Levi-Civita tensor has given rise to the $(-1)^{pq}$ factor, since each interchange of an index pair produces a minus sign. In getting to the third line, we have used the identity (1.241). In getting to the fourth line, we have used the basic property (1.240) of the multi-index Kronecker delta tensor. The upshot, therefore, is that applying the Hodge dual operation twice to a p-form ω in n dimensions, we get

$$**\omega = (-1)^{pq+t}\, \omega\,, \tag{1.251}$$

where $q = n - p$, and where t is the number of time directions (i.e. the number of negative eigenvalues of the metric tensor).

In cases where $pq+t$ is even, we shall have that $**\omega = \omega$, which means that the operator $*$ itself has eigenvalues ± 1. If the dimension n is even, say $n = 2m$, an m-form ω is mapped into another m-form by the Hodge $*$ operator, and so if $m^2 + t$ is even, we can make $\pm$ eigenstates under $*$, defined by

$$\omega_\pm = \tfrac{1}{2}(\omega \pm *\omega)\,. \tag{1.252}$$

these have the property that

$$*\omega_\pm = \pm\omega_\pm\,, \tag{1.253}$$

and they are known as self-dual or anti-self-dual forms respectively. This possibility therefore arises in Riemannian geometry (i.e. $t = 0$) in dimensions $n = 4, 8, 12, \ldots$. In pseudo-Riemannian geometry with a single time dimension (i.e. $t = 1$), (anti)-self-duality is instead possible in dimensions $n = 2, 6, 10, \ldots$.

The Hodge dual provides a nice way of taking the inner product of two p-forms. Suppose we have two p-forms, A and B in an n-dimensional manifold M. Defining $q = n - p$ as usual, we shall have

$$
\begin{aligned}
*A\wedge B &= \frac{1}{(p!)^2\, q!}\,\epsilon_{i_1\cdots i_q}{}^{j_1\cdots j_p}\, A_{j_1\cdots j_p}\, B_{k_1\cdots k_p}\, dx^{i_1}\wedge\cdots\wedge dx^{i_q}\wedge dx^{k_1}\wedge\cdots\wedge dx^{k_p}\\
&= \frac{(-1)^t}{(p!)^2\, q!}\,\epsilon_{i_1\cdots i_q}{}^{j_1\cdots j_p}\, A_{j_1\cdots j_p}\, B_{k_1\cdots k_p}\, \varepsilon^{i_1\cdots i_q k_1\cdots k_p}\, dx^1\wedge dx^2\wedge\cdots\wedge dx^n\\
&= \frac{(-1)^t}{(p!)^2\, q!}\,\epsilon_{i_1\cdots i_q}{}^{j_1\cdots j_p}\, A_{j_1\cdots j_p}\, B_{k_1\cdots k_p}\, \epsilon^{i_1\cdots i_q k_1\cdots k_p}\,\sqrt{|g|}\, dx^1\wedge dx^2\wedge\cdots\wedge dx^n\\
&= \frac{1}{p!}\, A_{j_1\cdots j_p}\, B_{k_1\cdots k_p}\, \delta^{k_1\cdots k_q}_{i_1\cdots i_q}\,\sqrt{|g|}\, dx^1\wedge dx^2\wedge\cdots\wedge dx^n\\
&= \frac{1}{p!}\, A_{i_1\cdots i_p}\, B^{i_1\cdots i_p}\,\sqrt{|g|}\, dx^1\wedge dx^2\wedge\cdots\wedge dx^n\,. \qquad (1.254)
\end{aligned}
$$

Thus we can write

$$*A\wedge B = \frac{1}{p!}\, A_{i_1\cdots i_p}\, B^{i_1\cdots i_p}\, {*\mathbb{1}}\,, \qquad (1.255)$$

where

$$*\mathbb{1} = \frac{1}{n!}\,\epsilon_{i_1\cdots i_n}\, dx^{i_1}\wedge\cdots\wedge dx^{i_n} = \sqrt{|g|}\, dx^1\wedge dx^2\wedge\cdots\wedge dx^n\,. \qquad (1.256)$$

Note that $*\mathbb{1}$, which is the Hodge dual of the constant 1, calculated using the standard rule (1.246) applied to a 0-form, is the *volume form.* For example, in Cartesian coordinates on Euclidean 2-space, where the metric is just $ds^2 = dx^2 + dy^2$, we would have $*\mathbb{1} = dx\wedge dy$, whilst in polar coordinates, where the metric is $ds^2 = dr^2 + r^2 d\theta^2$, we would have $*\mathbb{1} = r dr\wedge d\theta$. Thus equation (1.255) shows that $*A\wedge B$ is equal to $1/p!$ times the volume form, multiplied by the inner product

$$|A\cdot B| \equiv A_{i_1\cdots i_p}\, B^{i_1\cdots i_p} \qquad (1.257)$$

of the two p-forms A and B. The inner product is manifestly symmetric under the exchange of A and B, and so we have

$$*A\wedge B = *B\wedge A = \frac{1}{p!}\,|A\cdot B|\, {*\mathbb{1}}\,. \qquad (1.258)$$

Of course if the metric has all positive eigenvalues (i.e. $t=0$), then the inner product is positive semi-definite, in the sense that

$$|A\cdot A| \ge 0\,, \qquad (1.259)$$

with equality if and only if $A=0$.

1.14 The δ Operator and the Laplacian

1.14.1 The adjoint operator δ; covariant divergence

Let A and B be two p-forms. We may define the quantity (A, B) by

$$(A, B) \equiv \int_M *A \wedge B\,, \tag{1.260}$$

where, by (1.258), the integrand is the n-form proportional to the volume form times the inner product of A and B. Like the unintegrated inner product, it is the case that if the metric has all positive eigenvalues, then (A, B) is positive semi-definite, in the sense that

$$(A, A) \geq 0\,, \tag{1.261}$$

with equality if and only if A vanishes everywhere in M. Note that from (1.258) we also have that

$$(A, B) = (B, A)\,. \tag{1.262}$$

Suppose now we have a p-form ω and $(p-1)$-form ν. Using the definition (1.260) we may form the quantity $(\omega, d\nu)$. Let us assume that the n-manifold M has no boundary. By using Stokes' theorem, we can perform the following manipulation:

$$\begin{aligned}(\omega, d\nu) &= \int_M *\omega \wedge d\nu = (-1)^q \int_M d(*\omega \wedge \nu) - (-1)^q \int_M d*\omega \wedge \nu \\ &= (-1)^q \int_{\partial M} *\omega \wedge \nu - (-1)^q \int_M d*\omega \wedge \nu \\ &= (-1)^{q+1} \int_M d*\omega \wedge \nu = (-1)^{pq+p+t} \int_M *(*d*\omega) \wedge \nu \\ &= (-1)^{pq+p+t}\,(*d*\omega, \nu)\,,\end{aligned} \tag{1.263}$$

where as usual we have defined $q \equiv n - p$. Thus it is natural to define the *adjoint* of the exterior derivative, which is called δ, to be such that for any p-form ω and any $(p-1)$-form ν, we shall have

$$(\omega, d\nu) = (\delta\omega, \nu)\,, \tag{1.264}$$

with

$$\delta \equiv (-1)^{pq+p+t} *d* = (-1)^{np+t} *d*\,. \tag{1.265}$$

Of course from (1.262) we shall also have

$$(\nu, \delta\omega) = (d\nu, \omega)\,. \tag{1.266}$$

Note that using (1.265) and (1.251) we can immediately see that δ has the property that

$$\delta^2 = 0 \tag{1.267}$$

when acting on any p-form. Note that δ maps a p-form to a $(p-1)$-form.

We know that d maps a p-form ω to a $(p+1)$-form, and that the Hodge dual $*$ maps a p-form to an $(n-p)$-form in n dimensions. It is easy to see, therefore, that the operator $*d*$ applied to a p-form gives a $(p-1)$-form. What is the object $*d*\omega$? It is actually related to something very simple, namely the divergence of ω, with components $\nabla^k \omega_{k i_1 \cdots i_{p-1}}$. To show this is straightforward, although a little lengthy. For the sake of completeness, we shall give the derivation here. Those steps in the argument that are analogous to ones that have already been spelt out in previous derivations will be performed this time without further comment. We shall have

$$
\begin{aligned}
\omega &= \frac{1}{p!}\, \omega_{i_1\cdots i_p}\, dx^{i_1}\wedge\cdots\wedge dx^{i_p}\,, \\
*\omega &= \frac{1}{p!\,q!}\, \omega_{i_1\cdots i_p}\, \epsilon_{j_1\cdots j_q}{}^{i_1\cdots i_p}\, dx^{j_1}\wedge\cdots\wedge dx^{j_q}\,, \\
d{*}\omega &= \frac{1}{p!\,q!}\, \partial_k(\omega_{i_1\cdots i_p}\, \epsilon_{j_1\cdots j_q}{}^{i_1\cdots i_p})\, dx^k\wedge dx^{j_1}\wedge\cdots\wedge dx^{j_q}\,, \\
d{}\omega &= \frac{1}{p!\,q!\,(p-1)!}\, \partial_k(\omega_{i_1\cdots i_p}\, \epsilon_{j_1\cdots j_q}{}^{i_1\cdots i_p})\, \epsilon_{\ell_1\cdots\ell_{p-1}}{}^{k j_1\cdots j_q}\, dx^{\ell_1}\wedge\cdots\wedge dx^{\ell_{p-1}} \\
&= \frac{1}{p!\,q!\,(p-1)!}\, \partial_k(\omega^{i_1\cdots i_p}\, \epsilon_{j_1\cdots j_q i_1\cdots i_p})\, \epsilon_{\ell_1\cdots\ell_{p-1}}{}^{k j_1\cdots j_q}\, dx^{\ell_1}\wedge\cdots\wedge dx^{\ell_{p-1}} \\
&= \frac{(-1)^{pq}}{p!\,q!\,(p-1)!}\, \partial_k(\omega^{i_1\cdots i_p}\, \epsilon_{i_1\cdots i_p j_1\cdots j_q})\, \epsilon_{\ell_1\cdots\ell_{p-1}}{}^{k j_1\cdots j_q}\, dx^{\ell_1}\wedge\cdots\wedge dx^{\ell_{p-1}} \\
&= \frac{(-1)^{pq}}{p!\,q!\,(p-1)!}\, \partial_k(\omega^{i_1\cdots i_p}\, \sqrt{|g|}\, \varepsilon_{i_1\cdots i_p j_1\cdots j_q})\, \epsilon_{\ell_1\cdots\ell_{p-1}}{}^{k j_1\cdots j_q}\, dx^{\ell_1}\wedge\cdots\wedge dx^{\ell_{p-1}} \\
&= \frac{(-1)^{pq}}{p!\,q!\,(p-1)!}\, \partial_k(\omega^{i_1\cdots i_p}\, \sqrt{|g|})\, \varepsilon_{i_1\cdots i_p j_1\cdots j_q}\, \epsilon_{\ell_1\cdots\ell_{p-1}}{}^{k j_1\cdots j_q}\, dx^{\ell_1}\wedge\cdots\wedge dx^{\ell_{p-1}} \\
&= \frac{(-1)^{pq}}{p!\,q!\,(p-1)!}\, \frac{1}{\sqrt{|g|}}\, \partial_k(\omega^{i_1\cdots i_p}\, \sqrt{|g|})\, \epsilon_{i_1\cdots i_p j_1\cdots j_q}\, \epsilon_{\ell_1\cdots\ell_{p-1}}{}^{k j_1\cdots j_q}\, dx^{\ell_1}\wedge\cdots\wedge dx^{\ell_{p-1}}\,,
\end{aligned}
\tag{1.268}
$$

where the only new type of manipulation so far is to replace the Levi-Civita tensor $\epsilon_{i_1\cdots i_n}$ by $\sqrt{|g|}\, \varepsilon_{i_1\cdots i_n}$, take the Levi-Civita tensor density $\varepsilon_{i_1\cdots i_n}$ outside the partial derivative (which can be done since it has constant components ± 1 and 0), and then restore it to the Levi-Civita tensor by dividing out by $\sqrt{|g|}$ once it is outside the partial derivative. It is helpful at this point to define the object

$$
Y_k{}^{i_1\cdots i_p} \equiv \frac{1}{\sqrt{|g|}}\, \partial_k(\sqrt{|g|}\, \omega^{i_1\cdots i_p})\,,
\tag{1.269}
$$

which we will shortly be able to turn into something recognisable. Continuing to the next step that follows on from the last line in (1.268), we can write

$$
d{}\omega = \frac{(-1)^{pq}}{p!\,q!\,(p-1)!}\, Y^k{}_{i_1\cdots i_p}\, \epsilon^{i_1\cdots i_p}{}_{j_1\cdots j_q}\, \epsilon_{\ell_1\cdots\ell_{p-1}\,k}{}^{j_1\cdots j_q}\, dx^{\ell_1}\wedge\cdots dx^{\ell_{p-1}}
$$

$$
\begin{aligned}
&= \frac{(-1)^{pq+t}}{(p-1)!}\, Y^k{}_{i_1\cdots i_p}\, \delta^{i_1\cdots i_p}_{\ell_1\cdots\ell_{p-1}\,k}\, dx^{\ell_1}\wedge\cdots dx^{\ell_{p-1}} \\
&= \frac{(-1)^{pq+t}}{(p-1)!}\, Y^k{}_{\ell_1\cdots\ell_{p-1}\,k}\, dx^{\ell_1}\wedge\cdots dx^{\ell_{p-1}} \\
&= \frac{(-1)^{pq+p+1+t}}{(p-1)!}\, Y^k{}_{k\ell_1\cdots\ell_{p-1}}\, dx^{\ell_1}\wedge\cdots dx^{\ell_{p-1}}\,. \qquad (1.270)
\end{aligned}
$$

Now, we have

$$
\begin{aligned}
Y^k{}_{k\ell_1\cdots\ell_{p-1}} &= Y_k{}^{km_1\cdots m_{p-1}}\, g_{\ell_1 m_1}\cdots g_{\ell_{p-1}\, m_{p-1}} \\
&= \frac{1}{\sqrt{|g|}}\, \partial_k(\sqrt{|g|}\,\omega^{km_1\cdots m_{p-1}})\, g_{\ell_1 m_1}\cdots g_{\ell_{p-1}\, m_{p-1}} \\
&= (\nabla_k\, \omega^{km_1\cdots m_{p-1}})\, g_{\ell_1 m_1}\cdots g_{\ell_{p-1}\, m_{p-1}} \\
&= \nabla^k \omega_{k\ell_1\cdots\ell_{p-1}}\,, \qquad (1.271)
\end{aligned}
$$

where the step of passing to the third line involves using results derived in section 1.9, and the symmetry of $\Gamma^i{}_{jk}$ in its two lower indices. (A special case, for a 2-index antisymmetric tensor, was on Problem Sheet 2.)

Finally, we are approaching the bottom line, namely that we have found

$$
d{\omega} = \frac{(-1)^{pq+p+t+1}}{(p-1)!}\, \nabla^k \omega_{ki_1\cdots i_{p-1}}\, dx^{i_1}\wedge\cdots\wedge dx^{i_{p-1}}\,. \qquad (1.272)
$$

In other words, we have shown that the components of the $(p-1)$-form $*d{*\omega}$ are given by[7]

$$
(*d{*\omega})_{i_1\cdots i_{p-1}} = (-1)^{pq+p+t+1}\, \nabla^k \omega_{ki_1\cdots i_{p-1}}\,. \qquad (1.273)
$$

Comparing this with (1.265), we see that for any p-form ω, we shall have

$$
(\delta\omega)_{i_1\cdots i_{p-1}} = -\nabla^k \omega_{ki_1\cdots i_p}\,. \qquad (1.274)
$$

[7]Note that although this derivation may have seemed like a bit of a long song and dance, much of this was because, for pedagodgic reasons, all the logical steps have been spelt out. Additionally, we presented rather carefully the mechanism by which the partial derivative turned into a covariant divergence. We could have short-circuited quite a few of those steps by making the following argument: We know that the exterior derivative d maps a p-form to a $(p+1)$-form, and we know that the Hodge $*$ maps a p-form to an $(n-p)$-form. Therefore we *know* that $*d{*\omega}$ must be a $(p-1)$-form, and therefore that it must be an honest tensorial object. Thus, as soon as we saw the ∂_k appear in the expression for $d{*\omega}$, we know on the grounds of covariance, that we *must* be able to replace the partial derivative by a covariant one, since the answer must be covariant, so what else could it be? All we are doing by replacing ∂_k by ∇_k is making a "hidden" non-manifest covariance into an explicit manifest covariance. If we allow ourselves to make that replacement, we more quickly end up at the same conclusion.

1.14.2 The Laplacian

We have already met the covariant Laplacian operator that acts on scalars. Here, we give the generalisation to a Laplacian operator that acts on p-forms of any rank. It is defined by

$$\Delta \equiv d\delta + \delta d\,. \tag{1.275}$$

Since d maps p-forms to $(p+1)$-forms, and δ maps p-forms to $(p-1)$-forms, we see that each of the two terms in Δ maps a p-form back into a p-form, and thus so does Δ itself.

If we apply Δ to a scalar f, then, noting that $\delta f \equiv 0$ (since δf would be a (-1)-form, which doesn't exist), we shall have

$$\Delta f = \delta df = -\nabla^i \nabla_i f\,. \tag{1.276}$$

Thus when acting on scalars, Δ is the *negative* of what one commonly calls the Laplacian in more elementary contexts. It is actually rather natural to include the minus sign in the definition, because $\Delta = -\nabla^i \nabla_i$ is then a positive operator when acting on scalars, in the case that the metric has all positive eigenvalues.

In fact, more generally, we can see that Δ defined by (1.275) is a positive operator when acting on any p-form, in the case that the metric has all positive eigenvalues (i.e. $t = 0$). To see this, let ω be an arbitrary p-form, and assume that M is a compact n-manifold equipped with a positive-definite metric. Then we shall have

$$(\omega \Delta \omega) = (\omega, d\delta\omega) + (\omega, \delta d\omega) = (d\omega, d\omega) + (\delta\omega, \delta\omega)\,. \tag{1.277}$$

As noted previously, we have $(A, A) \geq 0$, with equality if and only if $A = 0$, and so we conclude that

$$(\omega, \Delta\omega) \geq 0\,, \tag{1.278}$$

with equality if and only if $\Delta\omega = 0$. A p-form ω that satisfies $\Delta\omega = 0$ is called an *harmonic* p-form. Furthermore, (1.277) shows that $\Delta\omega = 0$ if and only if

$$d\omega = 0\,, \qquad \delta\omega = 0\,. \tag{1.279}$$

We already met the notion of a *closed* p-form ω, as being one for which $d\omega = 0$. We can also introduce the notion of a *co-closed* p-form ω, as being one for which $\delta\omega = 0$. Thus we have seen that on a manifold without boundary, equipped with a positive-definite metric, a p-form is harmonic if and only if it is both closed and co-closed.

We have already seen that when acting on scalars f (i.e. 0-forms), the Laplacian operator is given by

$$\Delta f = -\Box f\,, \tag{1.280}$$

where we define

$$\Box \equiv \nabla^i \nabla_i \,. \tag{1.281}$$

It is straightforward to evaluate the Laplacian acting on forms of higher degree, by making use of the expressions (1.199) and (1.274) for the components of $d\omega$ and $\delta\omega$. For example, acting on a 1-form V, and on a 2-form ω, one finds

$$\begin{aligned}
(\Delta V)_i &= -\Box V_i + R_{ij}\, V^j \,, \\
(\Delta \omega)_{ij} &= -\Box \omega_{ij} - 2R_{ikj\ell}\, \omega^{k\ell} + R_{ik}\, \omega^k{}_j + R_{jk}\, \omega_i{}^k \,.
\end{aligned} \tag{1.282}$$

Note that the curvatures arise because terms in the expression for Δ give rise to commutators of covariant derivatives.

1.15 Spin connection and curvature 2-forms

When we introduced the notations of the covariant derivative, in section 1.9, and the Riemann tensor, in section 1.10, this was done in the framework of a choice of coordinate basis. We have already discussed the idea of using a non-coordinate basis for the tangent and co-tangent frames, and here we return to this, in order to introduce a different way of defining the connection and curvature. It is, in the end, equivalent to the coordinate-basis description, but it has various advantages, including (relative) computational simplicity.

We begin by "taking the square root" of the metric g_{ij}, by introducing a vielbein, which is a basis of 1-forms $e^a = e_i^a\, dx^i$, with the components e_i^a having the property

$$g_{ij} = \eta_{ab}\, e_i^a\, e_j^b \,. \tag{1.283}$$

Here the indices a are local-Lorentz indices, or tangent-space indices, and η_{ab} is a "flat" metric, with constant components. The language of "local-Lorentz" indices stems from the situation when the metric g_{ij} has Minkowskian signature (which is $(-,+,+,\ldots,+)$ in sensible conventions). The signature of η_{ab} must be the same as that of g_{ij}, so if we are working in general relativity with Minkowskian signature we will have

$$\eta_{ab} = \mathrm{diag}\,(-1,1,1,\ldots,1) \,. \tag{1.284}$$

If, on the other hand, we are working in a space with Euclidean signature $(+,+,\ldots,+)$, then η_{ab} will just equal the Kronecker delta, $\eta_{ab} = \delta_{ab}$, or in other words

$$\eta_{ab} = \mathrm{diag}\,(1,1,1,\ldots,1) \,. \tag{1.285}$$