Erratum/Remarks

1 Canonical transformations

We have shown that if we have a change of variables \((q, p) \rightarrow (Q, P)\) that leaves Hamilton’s equations unaltered,

\[
\dot{Q}_\alpha = \frac{\partial H}{\partial P_\alpha},
\]
\[
\dot{P}_\alpha = -\frac{\partial H}{\partial Q_\alpha},
\]

then the new variables have to obey canonical Poisson brackets:

\[
[Q_\alpha, Q_\beta]_{q,p} = 0, \quad [P_\alpha, P_\beta]_{q,p} = 0, \quad [Q_\alpha, P_\beta]_{q,p} = \delta_{\alpha\beta}.
\]

One of you raised the question: how do we prove the inverse implication—that is, that (3) is enough to ensure (1) and (2)?

We proved several things involving Poisson brackets, including:

1. For any function \(g(q,p)\) of the original canonical conjugate variables, one has

\[
\frac{dg}{dt} = [g, H]_{q,p}.
\]

2. If \((Q, P)\) obey canonical Poisson brackets as in (3), then for any function \(f(Q,P)\) one has

\[
[Q_\alpha, f(Q,P)]_{q,p} = \frac{\partial f}{\partial P_\alpha}, \quad [P_\alpha, f(Q,P)]_{q,p} = -\frac{\partial f}{\partial Q_\alpha}.
\]

Combining these two facts in order, if \((Q, P)\) obey canonical Poisson brackets we have

\[
\dot{Q}_\alpha = [Q_\alpha, H]_{q,p} = \frac{\partial H}{\partial P_\alpha},
\]
\[
\dot{P}_\alpha = [P_\alpha, H]_{q,p} = -\frac{\partial H}{\partial Q_\alpha},
\]

as desired (the \(Q\)'s and the \(P\)'s are functions of \((q, p)\), and can thus play the role of \(g\) in (4)).

2 Poincaré recurrence theorem

- As one of you noticed, we have to assume that the Hamiltonian does not depend explicitly on time, otherwise we are not allowed to manipulate the phase flow \(g_t\) like we did, for instance in

\[
g_{-t}(g_k(U) \cap g_t(U)) = g_{k-t}(U) \cap U.
\]
If the Hamiltonian depends on time, the phase flow $g_t$ depends both on the duration $t$ of the time-evolution, as well as on the starting time $t_0$.

$$g_t \rightarrow g_{t,t_0}$$

which breaks the nice composition rule (8).

- At the final steps of the theorem, we had

$$g_{k-l}(U) \cap U \neq \emptyset .$$

We than said, pick $a \in g_{k-l}(U) \cap U$, and we are done. In fact, this is wrong. What (10) is telling us is that part of $g_{k-l}(U)$ overlaps with $U$. Recall that all points of $g_{k-l}(U)$ come from $U$—i.e., they were in $U$ at $t = 0$, and at $t = k - l$ they are in $g_{k-l}(U)$. Eq. (10) is thus telling us that some points of $U$ go back to $U$ after $t = k - l$. There is an (in fact, many) $a \in U$, that after $t = k - l$ ends up in $g_{k-l}(U) \cap U$:

$$g_{k-l}(a) \in g_{k-l}(U) \cap U \subset U .$$

Or equivalently, pick any point $b \in g_{k-l}(U) \cap U$ and evolve it back in time for $t = k - l$,

$$a \equiv g_{-(k-l)}(b) .$$

Such an $a$ is in $U$ (because $b$ is in $g_{k-l}(U)$), and after $t = k - l$ is back to $U$ (because $b = g_{k-l}(a)$ is in $U$).